

IB DIPLOMA MATHEMATICS AI

HL
PAPER 2
2021 - 2025

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1 - (MATHAI/20_HL_Winter_2021_Q5) - Algebra-functions - Roots

[Maximum mark: 13]

(a) Let $z = 1 - i$.(i) Plot the position of z on an Argand Diagram.(ii) Express z in the form $z = ae^{ib}$, where $a, b \in \mathbb{R}$, giving the exact value of a and the exact value of b .

[3]

(b) Let $w_1 = e^{ix}$ and $w_2 = e^{i(x-\frac{\pi}{2})}$, where $x \in \mathbb{R}$.(i) Find $w_1 + w_2$ in the form $e^{ix}(c + id)$.(ii) Hence find $\operatorname{Re}(w_1 + w_2)$ in the form $A \cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \leq \frac{\pi}{2}$.

[6]

The current, I , in an AC circuit can be modelled by the equation $I = a \cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_A and I_B . The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_A	12 amps	0
I_B	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_T can be expressed as $I_A + I_B$.

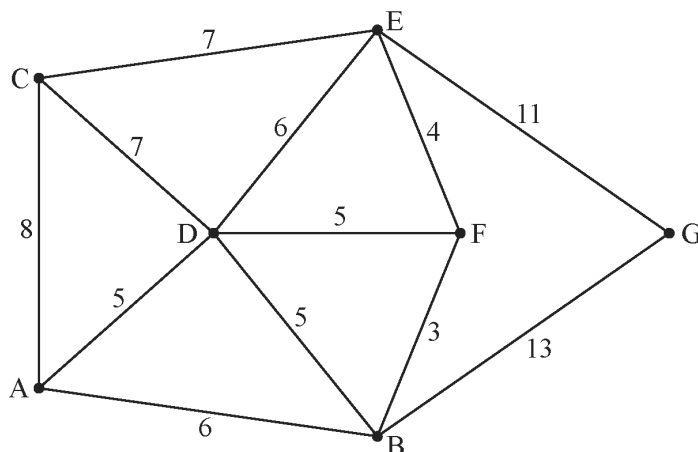
(c) (i) Find the maximum value of I_T .(ii) Find the phase shift of I_T .

[4]

2 - (MATHAI/21_HL_Summer_2023_Q4) - Algebra-functions - Roots

[Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



- (a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer.

[2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	<i>a</i>	<i>b</i>
D	5	5	7		6	5	<i>c</i>
E	11	7	7	6		4	11
F	9	3	<i>a</i>	5	4		<i>d</i>
G	19	13	<i>b</i>	<i>c</i>	11	<i>d</i>	

(b) Find the value of

(i) a ;

(ii) b ;

(iii) c ;

(iv) d . [2]

(c) Use the nearest neighbour algorithm, starting at vertex G , to find an upper bound for the travelling salesman problem. [3]

(d) (i) Sketch a minimum spanning tree for the subgraph with vertices A, B, C, D, E, F .

(ii) Write down the total weight of the minimum spanning tree. [4]

(e) Hence find a lower bound for the travelling salesman problem. [2]

(f) Explain one way in which an improved lower bound could be found. [1]

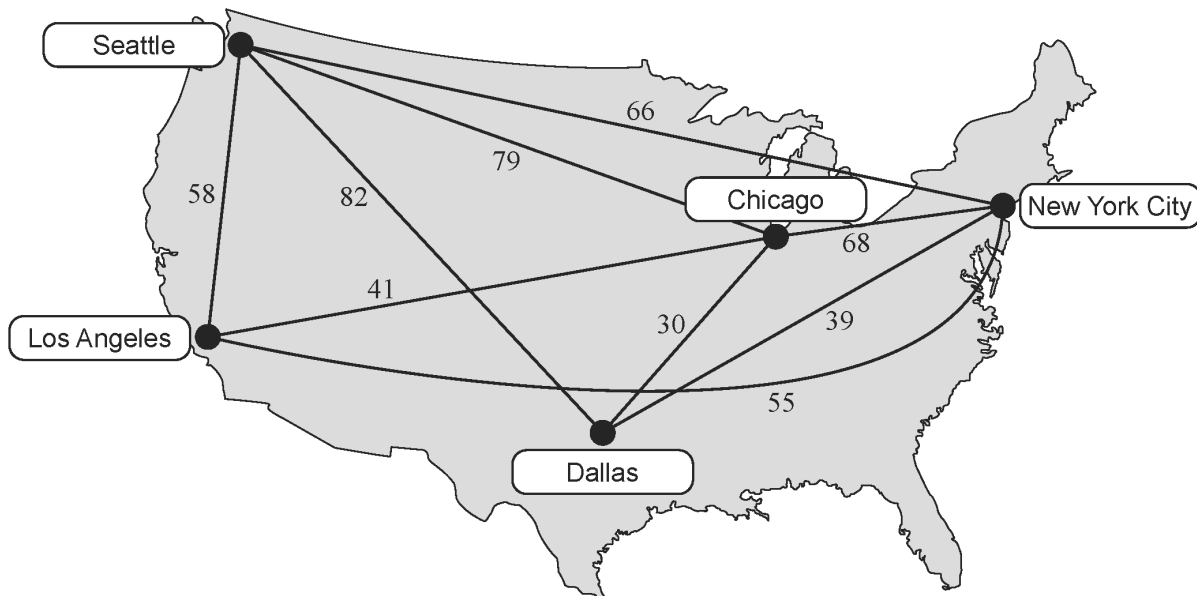
It is found that the optimum solution starting at A is actually $A-C-E-G-B-F-D-A$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

3 - (MATHAI/22_HL_Summer_2023_Q4) - Algebra-functions - Roots

[Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars (\$) between cities.



- (a) Explain why the graph can be described as “connected”, but not “complete”. [2]
- (b) Find a minimum spanning tree for the graph using Kruskal's algorithm. [3]
State clearly the order in which your edges are added, and draw the tree obtained.
- (c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing \$26. He updates the graph to show this.

- (d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c). [3]
State clearly the order in which you are adding the vertices.

- (e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.
- (ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]
- (f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]
- (g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

4 - (MATHAI/2_HL_Winter_2023_Q3) - Algebra-functions - Roots

[Maximum mark: 16]

Tiffany wants to buy a house for a price of 285 000 US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to 15 % of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a 4 % nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

- (a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.
- (ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]
- (b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays 1300 USD each month.

- (c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than 1300 USD.

- (d) Determine the amount of Tiffany's final payment, to two decimal places. [4]
- (e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

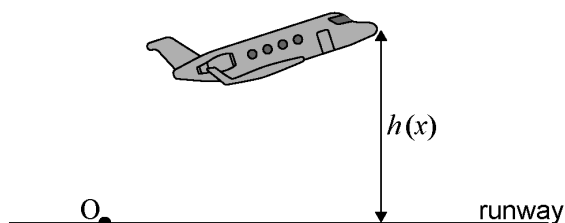
5 - (MATHAI/2_HL_Winter_2023_Q4) - Algebra-functions - Roots, Differentiation

[Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1 + 150e^{-0.07x}} - 0.06, \quad x \geq 0.$$

diagram not to scale



- (a) (i) Find $h(0)$.
(ii) Interpret this value in terms of the context. [2]
- (b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.
(ii) Interpret this value in terms of the context. [2]
- (c) Find $h'(x)$ in terms of x . [4]
- A safety regulation recommends that $h'(x)$ never exceed 0.2.
- (d) Given that this plane flies a distance of at least 200 km horizontally from point O, determine whether the plane is following this safety regulation. [4]

6 - (MATHAI/22_HL_Summer_2024_Q4) - Algebra-functions - Roots

[Maximum mark: 12]

A hygiene inspector lives in Town A and must visit restaurants in five towns (B–F), before returning to A. The inspector must not repeat any of the towns. The distances, in km, between the six towns are shown in the table.

	A	B	C	D	E	F
A		31	28	26	22	23
B	31		25	20	27	25
C	28	25		19	22	24
D	26	20	19		21	22
E	22	27	22	21		24
F	23	25	24	22	24	

- (a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the length of the journey the inspector must take. State the order in which the towns are to be visited. [4]
- (b) By deleting A, use Prim's algorithm starting at B to find a lower bound for the length of the inspector's journey. [5]
- (c) By considering the minimum spanning tree found in part (b), determine whether the journey given by this lower bound is an achievable solution. [3]

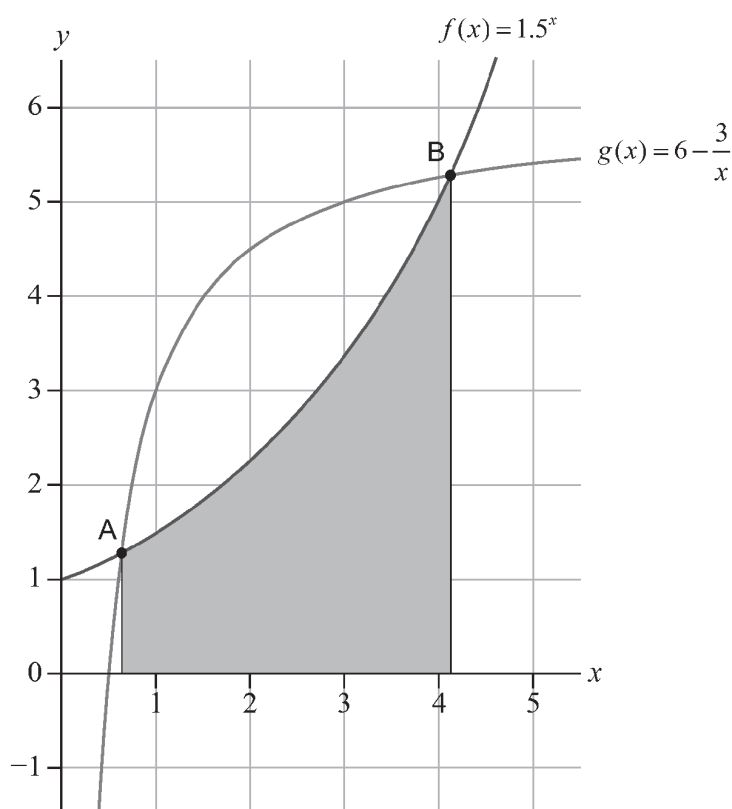
7 - (MATHAI/22_HL_Summer_2025_Q2) - Algebra-functions - Roots, Integration, Differentiation

[Maximum mark: 12]

The diagram shows part of the graphs of the functions

$$f(x) = 1.5^x \quad x \geq 0$$

$$g(x) = 6 - \frac{3}{x} \quad x > 0.$$



(a) Solve $f(x) = g(x)$. [3]

(b) (i) Write down the integral that represents the area of the shaded region.

(ii) Calculate the area of this shaded region.

(iii) Hence, or otherwise, calculate the area of the region enclosed between the curves $y = f(x)$ and $y = g(x)$. [6]

The tangent to the graph of $y = f(x)$ is parallel to the tangent to the graph of $y = g(x)$ at $x = k$.

(c) Find the value of k . [3]

8 - (MATHAI/22_HL_Summer_2025_Q3) - Geometry, Algebra-functions - Roots, Differentiation

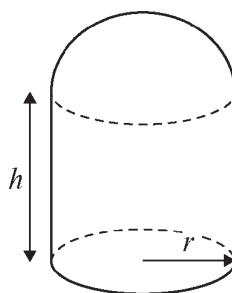
[Maximum mark: 16]

Ju Shen designs a plastic cover, in the shape of a cylinder combined with a hemisphere on top, as shown in the diagram.

The plastic used to make the cover forms the curved surface of both the hemisphere and the cylinder; there is no bottom to the cover, however it stands on a flat horizontal surface.

Let the height of the cylinder be h cm and the internal radius of its base be r cm.

diagram not to scale



- (a) Find an expression for the total internal surface area, A cm², of the cover in terms of r and h . [2]
- (b) Show that the total volume, V cm³, enclosed by the cover and the horizontal surface is given by the expression

$$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}. \quad [2]$$

The total volume enclosed by the cover is 10 000 cm³.

- (c) Hence show that $h = \frac{30\,000 - 2\pi r^3}{3\pi r^2}$. [2]

Ju Shen uses the total internal surface area to model the amount of plastic used to construct the cover.

- (d) Show that A is given by the expression

$$A = \frac{2\pi r^2}{3} + \frac{20\,000}{r}. \quad [2]$$

Ju Shen wants to use the minimum amount of plastic in the construction of the cover.

- (e) Find an expression for $\frac{dA}{dr}$. [3]
- (f) Find the value of r and the value of h that minimizes the use of plastic. [4]
- (g) By interpreting your answer to part (f), suggest the best shape for Ju Shen's plastic cover. [1]

9 - (MATHAI/22_HL_Summer_2025_Q7) - Matrices, Probability, Algebra-functions - Roots

[Maximum mark: 17]

On any given day, the probability that Emlyn charges his phone depends only on whether he charged it the previous day.

If he charged his phone the previous day, the probability he charges it today is 0.4.

If he did not charge his phone the previous day, the probability he charges it today is p .

On day n this can be represented using the vector $\mathbf{v}_n$ where

$$\mathbf{v}_n = \begin{pmatrix} \text{probability that Emlyn charges his phone on day } n \\ \text{probability that Emlyn does not charge his phone on day } n \end{pmatrix}$$

A Markov chain model is formed where

$$\mathbf{v}_{n+1} = \mathbf{M}\mathbf{v}_n.$$

Matrix $\mathbf{M}$ is of the form $\begin{pmatrix} a & p \\ b & 1-p \end{pmatrix}$.

(a) Write down the value of

(i) a .

(ii) b . [2]

(b) On day zero Emlyn charges his phone. Find the probability

(i) that Emlyn charges his phone on all days from $n = 1$ to $n = 4$.

(ii) that Emlyn charges his phone on day 4, when $p = 0.7$. [5]

(c) Demonstrate that, for all values of p , one eigenvector of $\mathbf{M}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and hence state the associated eigenvalue. [4]

(d) Find, in terms of p , the steady state probability that Emlyn charges his phone on a given day. [4]

In the long term, Emlyn wants to charge his phone on at least 60% of days.

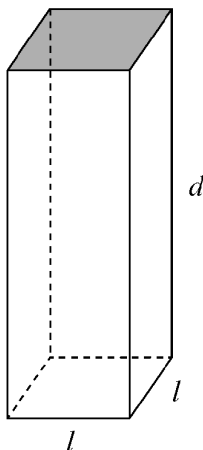
(e) Find the minimum value of p required for this to occur. [2]

10 - (MATHAI/23_HL_Summer_2025_Q1) - Geometry, Algebra-functions - Roots, Differentiation

[Maximum mark: 20]

Kailash manufactures drink containers in the shape of a cuboid. The container has a square top and a square base of length, l cm. Its height, d cm, is three times the length of the base.

diagram not to scale



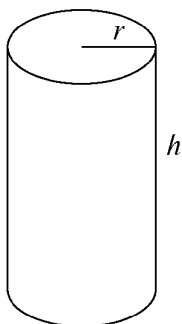
- (a) Write down an expression for d in terms of l . [1]

The container can hold 375 cm^3 of drink.

- (b) Find the value of l and d . [3]
- (c) Calculate the total external surface area of the container. [3]

To reduce environmental impact, Kailash is trying to minimize the amount of material needed for the production of the 375 cm^3 container.

He is willing to change the shape to a cylinder with radius $r \text{ cm}$, and height $h \text{ cm}$, as shown below.



The cylindrical container of drink must also hold 375 cm^3 .

- (d) Find an expression for the height, h , of the container in terms of r . [2]

Let the total external surface area be $A \text{ cm}^2$.

- (e) Show that $A = 2\pi r^2 + \frac{750}{r}$. [2]

- (f) Find $\frac{dA}{dr}$. [3]

- (g) Hence or otherwise

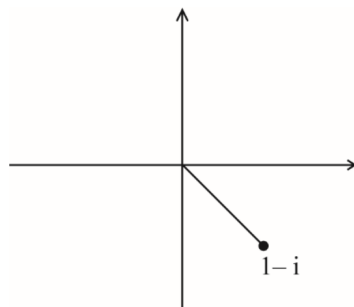
- (i) find the value of r that will minimize A .
(ii) find the minimum value of A needed for the cylinder. [3]

- (h) (i) Find $\frac{d^2A}{dr^2}$.
(ii) Hence determine whether the graph of A is concave-up or concave-down for $r > 0$. Justify your answer. [3]

ANSWERS

1 - (MATHAI/20_HL_Winter_2021_Q5) - Algebra-functions - Roots

(a) (i)



A1

(ii) $z = \sqrt{2}e^{\frac{i\pi}{4}}$

A1A1

Note: Accept an argument of $\frac{7\pi}{4}$. Do NOT accept answers that are not exact.

[3 marks]

$$\begin{aligned} \text{(b) (i)} \quad w_1 + w_2 &= e^{ix} + e^{i\left(x - \frac{\pi}{2}\right)} \\ &= e^{ix} \left(1 + e^{-\frac{i\pi}{2}} \right) \\ &= e^{ix}(1 - i) \end{aligned}$$

(M1)

A1

$$\begin{aligned} \text{(ii)} \quad w_1 + w_2 &= e^{ix} \times \sqrt{2}e^{\frac{i\pi}{4}} \\ &= \sqrt{2}e^{i\left(x - \frac{\pi}{4}\right)} \end{aligned}$$

M1

attempt extract real part using cis form

(A1)

(M1)

$$\operatorname{Re}(w_1 + w_2) = \sqrt{2} \cos\left(x - \frac{\pi}{4}\right) \quad \text{OR} \quad 1.4142... \cos(x - 0.785398...)$$

A1

[6 marks]

$$\text{(c) (i)} \quad I_t = 12 \cos(bt) + 12 \cos\left(bt - \frac{\pi}{2}\right)$$

(M1)

$$I_t = 12 \operatorname{Re} \left(e^{ibt} + e^{i\left(bt - \frac{\pi}{2}\right)} \right)$$

(M1)

$$I_t = 12\sqrt{2} \cos\left(bt - \frac{\pi}{4}\right)$$

$$\max = 12\sqrt{2} \quad (=17.0)$$

A1

$$\text{(ii)} \quad \text{phase shift} = \frac{\pi}{4} (=0.785)$$

A1

[4 marks]

Total: [13 marks]

2 - (MATHAI/21_HL_Summer_2023_Q4) - Algebra-functions - Roots

- (a) there are more than two vertices with odd degree **R1**
 so it is not possible to travel along each road exactly once **A1**

Note: Do not award **R0A1**.

Award **R1** for "There are 4 vertices with odd degree".

[2 marks]

- (b) $a = 11, b = 18, c = 17, d = 15$ **A2**

Note: Award **A1** for any one correct, **A2** for all four correct.

[2 marks]

- (c) attempt to use nearest neighbour algorithm **(M1)**

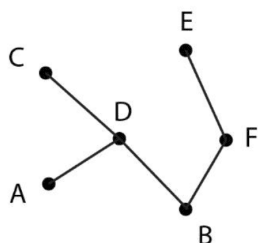
Note: Award **M1** for first 3 vertices correct or 11, 4, 3 seen.

G-E-F-B-D-A-C(-E)-G **OR** $11+4+3+5+5+8$ + their b **(A1)**

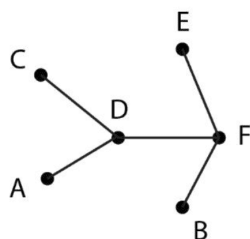
upper bound = 54 (km) **A1**

[3 marks]

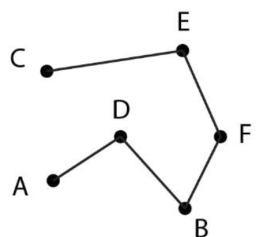
- (d) (i) a diagram of **any** spanning tree of the subgraph ABCDEF (A1)
 attempt at Kruskal's algorithm or Prim's algorithm (M1)
 e.g. edges BF (3), EF (4) and an edge of length 5 listed or seen in any
 spanning tree



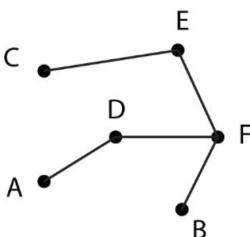
OR



OR



OR



A1

- (ii) 24 (km)

A1

Note: *FT* from their sketch, only if it is a spanning tree. It is not required to see the edge lengths on the sketch, since they are given in the question.

[4 marks]

- (e) adding vertex G's two shortest edges to their part (d)(ii)

(M1)

$$24 + 11 + 13$$

$$= 48$$

A1**[2 marks]**

- (f) try removing a different vertex

A1**[1 mark]**

- (g) recognize 7 edges in optimum route

(M1)

Note: Award **M1** for a total length of 52 seen.

subtracting $0.5 \times$ edges from 52**(M1)**

$$52 - 7 \times 0.5$$

$$= 48.5 \text{ (km)}$$

A1**[3 marks]**

3 - (MATHAI/22_HL_Summer_2023_Q4) - Algebra-functions - Roots

- (a) any city can be travelled to or from any other city (so is connected)

R1**EITHER**

but there is no direct flight between Los Angeles and Dallas (for example)

R1**OR**

but not every vertex has degree 4

R1

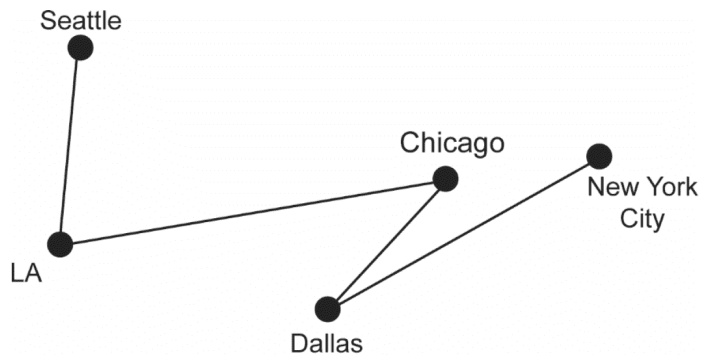
Note: Accept equivalent statements for the cities being connected and the graph not being complete.

[2 marks]

- (b) edge CD selected first
DN,
CL,
LS

M1**A1**

Note: Award marks if the answers are written as sums in the correct order.
M1 if 30 is seen first, **A1** for $30 + 39 + 41 + 58$.

**A1**

Note: The final **A1** can be awarded independently. Award **M0A0A1** for a correct MST graph with no other working. Award **M1A0A1** if Prim's algorithm is seen to be used correctly with CD first.

[3 marks]

- (c) $2 \times \text{MST weight}$
= \$336

(M1)**A1**

Note: Allow any integer multiple (>1) of MST weight for **M1**, and if correctly calculated, award **M1A1**.

[2 marks]

- (d) attempt at nearest neighbour algorithm
order is $LA \rightarrow D \rightarrow C \rightarrow NYC \rightarrow S \rightarrow LA$

M1**A1**

Note: Award **M1** for a route that begins with LA and then D, this includes seeing 26 as the first value in a sum.
Award **A1** if $26 + 30 + 68 + 66 + 58$ seen in order.

Note: Award **M1A0** for an incorrect first nearest neighbour proceeding 'correctly' to the next vertex. For example, LA to C and then C to D.

upper bound is $(26 + 30 + 68 + 66 + 58 =) \248

A1

Note: Award **M1A0** for correct nearest neighbour algorithm starting from a vertex other than LA. Condone the correct tour written backwards i.e.
 $58 + 66 + 68 + 30 + 26 = 248$

[3 marks]

- (e) (i) attempt to find MST of L, N, D and S **(M1)**
 by deleting C, Kruskal gives MST for the remainder as LD, DN, LS
 weight 123 **(A1)**
 (lower bound is therefore $123 + (30 + 41) =$ \$194 **A1**

Note: Award **(M1)** for a graph or list of edges that does not include C.
 Award **(A1)** if $26 + 39 + 58$ seen in any order.

- (ii) by deleting S, Kruskal gives MST for the remainder as LD, DC, DN
 weight 95 **(A1)**
 (lower bound is therefore $95 + (58 + 66) =$ \$219 **A1**

Note: Award **(A1)** if $26 + 30 + 39$ seen in any order.

[5 marks]

- (f) $219 \leq C \leq 248$ **A1A1**

Note: Award **A1** for $219 \leq C$ and **A1** for $C \leq 248$. Award at most **A1A0** for $219 < C < 248$.
FT for their values from part (e) if higher value from (e)(i) and (e)(ii) used for the lower bound, and part (d) for the upper.

[2 marks]

- (g) any valid tour, within their interval from part (f), from any starting point **OR**
 any valid tour that starts and finishes at N **(M1)**
 valid tour starting point N **AND** within their interval **A1**
 e.g. NDCLSN (weight 234)

Note: If part (f) not correct, **only** award **A1FT** if their valid tour begins and ends at N **AND** lies within **BOTH** their interval (including if one-sided) in part (f) **AND** $219 \leq C \leq 248$.
 If no response in the form of an interval seen in part (f) then award **M1A0** for a valid tour beginning and ending at N **AND** within $219 \leq C \leq 248$.

[2 marks]

4 - (MATHAI/2_HL_Winter_2023_Q3) - Algebra-functions - Roots

- (a) (i) attempt to find 15% or 85% of 285000 (M1)
 285000×0.85
 242250 (USD) A1

Note: Do not award A1 if answer is not given exact.

- (ii) $N = 360$
 $I\% = 4$
 $PV = (\pm) 242250$
 $FV = 0$
 $P/Y = 12$
 $C/Y = 12$ (M1)(A1)

Note: Award M1 for an attempt to use a financial app in their technology with at least two entries seen, award A1 for all entries correct.

(PMT =) 1156.54 (USD) A1

Note: Do not award final A1 if answer is not given to 2 dp.

[5 marks]

(b) 1156.54×360
416354 (USD)

(M1)
A1

Note: Do not award **A1** if answer is not given to the nearest dollar, unless already penalized in part (a)(ii).

[2 marks]

(c) $I\% = 4$
 $PV = (\pm) 242\,250$
 $PMT = (\mp) 1300$
 $FV = 0$
 $P/Y = 12$
 $C/Y = 12$

(A1)

Note: Award **A1** for $PMT = (\mp) 1300$ seen.

$(N \Rightarrow) 292$

A1

[2 marks]

(d) **METHOD 1**
 $N = 291$
 $I\% = 4$
 $PV = (\pm) 242\,250$
 $PMT = (\mp) 1300$
 $P/Y = 12$
 $F/Y = 12$

(A1)

Note: Award **A1** for $N = 291$ seen.

$(FV \Rightarrow) 871.91 \text{ (871.908...)}$

A1

valid attempt to find interest in final month (e.g. $N = 1$ OR $PV = 871.91$)

(M1)

$N = 1$
 $I\% = 4$
 $PV = 871.91 \text{ (871.908...)}$
 $FV = 0$
 $P/Y = 12$
 $F/Y = 12$

$(PMT \Rightarrow) 874.82 \text{ (USD)}$

A1

Note: Do not award **A1** if answer is not given correct to 2dp, unless already penalized previously.

METHOD 2

$$N = 292$$

$$I\% = 4$$

$$PV = (\pm) 242\,250$$

$$PMT = (\mp) 1300$$

$$P/Y = 12$$

$$F/Y = 12$$

(A1)**Note:** Award **A1** for $N = 292$ seen.

$$(FV =) 425.185\dots$$

A1

$$1300 - 425.185\dots$$

(A1)

$$(PMT =) 874.82 \text{ (USD)}$$

A1**Note:** Accept 874.81. Do not award **A1** if answer is not given correct to 2dp, unless already penalized previously.**[4 marks]**

(e) $291 \times 1300 + 874.82$

(M1)

$$379174.82$$

attempt to find difference between their value and their part (b)

(M1)

$$(416354 - 379174.82)$$

$$37179 \text{ (USD)}$$

A1**Note:** Accept 37180 (USD) from using the 2 dp. answer from part (b). Do not penalize for not rounding to nearest dollar if this has already been penalized in part (b).**[3 marks]****5 - (MATHAI/2_HL_Winter_2023_Q4) - Algebra-functions - Roots, Differentiation**

(a) (i) $h(0) = 0.00623 \text{ (km)} (= 0.00622517)$

A1(ii) this is the height of the nose of the plane (above the runway),
when the plane is on the runway**A1****[2 marks]**

(b) (i) $y = 9.94$

A1**Note:** Accept $h = 9.94$.(ii) **EITHER**

this is the height that the (nose of the) plane approaches (but does not reach)

A1**OR**

this is the maximum possible height of the (nose of the) plane

A1**OR**

the (nose of the) plane does not exceed this height

A1**[2 marks]**

(c) **METHOD 1 (chain rule)**

$$h(x) = 10(1 + 150e^{-0.07x})^{-1} - 0.06$$

(M1)

$$\text{find } h'(x) = -10(1 + 150e^{-0.07x})^{-2} \times 150e^{-0.07x} \times -0.07$$

A1M1A1

$$\left(= \frac{105e^{-0.07x}}{(1 + 150e^{-0.07x})^2} \right)$$

Note: Award **A1** for correct first term $(-10(1 + 150e^{-0.07x})^{-2})$, **M1** for attempt to use the chain rule, **A1** for correct use of chain rule $(\times 150e^{-0.07x} \times -0.07)$. Award at most **A1M1A0** if additional terms are seen. The answer is not required to be simplified beyond what is shown in the markscheme.

METHOD 2 (quotient rule)

$$\frac{(1 + 150e^{-0.007x})(0) - 10(150e^{-0.007x} \times -0.007)}{(1 + 150e^{-0.007x})^2}$$

M1A1

Note: Award **M1** for attempt to use quotient rule, **A1** for correct use.

$$= \frac{-10(150e^{-0.007x} \times -0.007)}{(1 + 150e^{-0.007x})^2} \quad \left(= \frac{105e^{-0.07x}}{(1 + 150e^{-0.07x})^2} \right)$$

A1A1

Note: Award **A1** for correct numerator and **A1** for correct denominator.

[4 marks](d) evidence of a graph of $h'(x)$ **(M1)**maximum at $x = 71.6$ ($= 71.58051\dots$)**(A1)**

$$h'(71.58051\dots) = 0.175$$

A1

maximum gradient is less than 0.2

and hence the regulation is being followed

A1**[4 marks]**

6 - (MATHAI/22_HL_Summer_2024_Q4) - Algebra-functions - Roots

(a) AEDCFBA

A1A1

Note: Award **A1** for AE at start, **A1** for correct completed route.

attempt to find the length of their route

length $22 + 21 + 19 + 24 + 25 + 31$ $= 142$ (km)

(M1)

A1

Note: Award **A1A0M1A0** for omitted final edge and their sum.

[4 marks]

(b) attempt to form MST without vertex A

(M1)

Note: Exactly 4 edges that form a spanning tree are required.

BD DC DE DF OR 20, 19, 21, 22 seen in that order

A1

Note: Award **M1A0** for diagram of MST.

attempt to reconnect vertex A (one edge is sufficient)

(M1)

reconnecting A: AE (22) and AF (23)

(A1)

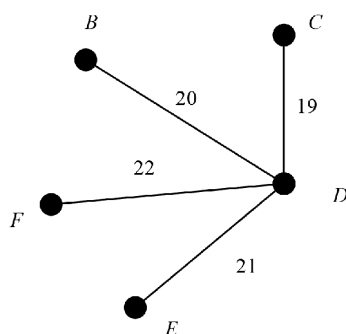
lower bound: $20 + 19 + 21 + 22 + 22 + 23$ $= 127$

A1

Note: If 127 seen, unsupported or without the explicit evidence of Prim's algorithm, award **M1A0M1A1A1**.

[5 marks]

(c)



(A1)

Note: Condone the omission of the weights from their diagram.
The diagram may include A with its two edges.

correct reasoning based on lack of cycle (once A is reattached)

R1

e.g. edges BD and CD would be repeated

this lower bound is not achievable (in this way)

A1

Note: Do not award **R0A1**.

[3 marks]

7 - (MATHAI/22_HL_Summer_2025_Q2) - Algebra-functions - Roots, Integration, Differentiation

(a) 0.638 (0.637623...) and 4.10 (4.09811...) **A2A1****Note:** Award **A2** for one correct answer, accept 4.1. Award **A2A0** if coordinate pair is given.**[3 marks]**(b) (i) $\int_{0.638}^{4.1} 1.5^x dx$ OR $\int_{0.638}^{4.1} f(x) dx$ **A1A1****Note:** **A1** for function and dx , **A1** for limits. Award **A1** for $\int_{0.638}^{4.1} y dx$.

Condone the use of modulus within the integrand.

(ii) 9.81 (9.80788...) (from 3sf)

9.80 (9.79844...) (from full)

A2**Note:** Accept 9.8 for **A2**.

(iii) correct expression for area

(A1)

$$\int_{0.638}^{4.1} g(x) - f(x) dx \text{ OR } \int_{0.638}^{4.1} 6 - \frac{3}{x} - 1.5^x dx$$

5.38 (5.38290...)

A1**Note:** Accept 5.39**[6 marks]**(c) $f'(x) = g'(x)$ **(M1)**correct graphs of $f'(x)$ and $g'(x)$ **(M1)****Note:** Award **M1** if $\frac{3}{x^2}$ seen.

(k =) 1.86 (1.86406...)

A1**[3 marks]****Total [12 marks]**

8 - (MATHAI/22_HL_Summer_2025_Q3) - Geometry, Algebra-functions - Roots, Differentiation

(a) $(A =) 2\pi r^2 + 2\pi r h$ A1A1

Note: Award **A1** for either term correct (even if unsimplified) and **A1** for all correct (including simplification and no extra term).

[2 marks]

(b) $V = \frac{2\pi r^3}{3} + \pi r^2 h$ A1A1

$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}$ AG

Note: Award **A1** for one correct term, **A1** for both and completion to required form.

[2 marks]

(c) 10000 substituted into (b) M1
correct rearrangement with at least one line of intermediate working leading to the given value.

M1

$h = \frac{30000 - 2\pi r^3}{3\pi r^2}$ AG

[2 marks]

(d) h substituted into their A M1

$(A =) 2\pi r^2 + 2\pi r \left(\frac{30000 - 2\pi r^3}{3\pi r^2} \right)$

further consistent rearrangement that leads to **given** A A1

[2 marks]

(e) $\frac{4\pi r}{3} - \frac{20000}{r^2}$ A1A1A1

Note: Award **A1** for $\frac{4\pi r}{3}$, **A1** for -20000 seen, and **A1** for a correct index for r in second term.

[3 marks]

(f) $\frac{dA}{dr} = 0$ (M1)

EITHER

rearrangement leading to $r^3 = \frac{15000}{\pi}$ (M1)

OR

graph of $\frac{dA}{dr}$ (M1)

THEN

$r = 16.8$ (16.8389...) OR $\sqrt[3]{\frac{15000}{\pi}}$ A1

Note: Accept correct graph of Area function leading to correct minimum value for r for **M1M1A1**.

$h = 0$ A1

[4 marks]

(g) hemisphere A1

Note: Do not **FT** from incorrect answers to part (f).

[1 mark]

Total [16 marks]

9 - (MATHAI/22_HL_Summer_2025_Q7) - Matrices, Probability, Algebra-functions - Roots

- (a) (i) 0.4 A1
 (ii) 0.6 A1

[2 marks]

- (b) (i) $0.4^4 = 0.0256$ (M1)A1
 (ii) considering $M^4 = \begin{pmatrix} 0.5422 & 0.5341 \\ 0.4578 & 0.4659 \end{pmatrix}$ (M1)(A1)

required probability is therefore 0.542

A1**[5 marks]**

- (c) attempting Mv with given eigenvector M1

$$\begin{pmatrix} 0.4 & p \\ 0.6 & 1-p \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

correct matrix multiplication

$$\begin{pmatrix} 0.4-p \\ -0.4+p \end{pmatrix}$$

A1

writing as a multiple of given vector

$$= (0.4 - p) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(A1)therefore the eigenvalue is $0.4 - p$ A1**[4 marks]**

- (d) consider eigenvector with eigenvalue 1 $Mv = v$ (M1)

$$\begin{pmatrix} 0.4 & p \\ 0.6 & 1-p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

turn into equations:

$$0.4x + py = x \quad \text{OR} \quad 0.6x + (1-p)y = y$$

A1also use $x + y = 1$ A1

$$\text{then } x + \frac{0.6x}{p} = 1$$

$$x = \frac{p}{p+0.6}$$

A1**[4 marks]**

- (e) graphical, inequality or equality approach to solve $\frac{p}{p+0.6} \geq 0.6$ (M1)

$$p = 0.9$$

A1**[2 marks]****Total [17 marks]**

10 - (MATHAI/23_HL_Summer_2025_Q1) - Geometry, Algebra-functions - Roots, Differentiation

(a) $(d=)3l$ A1

[1 mark]

(b) substitution into volume of cuboid formula $V = l^2 \times 3l (= 375)$ (M1)

attempting to solve their $3l^3 = 375$ (M1)

$$l^3 = 125 \text{ OR } l = \sqrt[3]{125} \text{ OR } l^3 = \frac{375}{3}$$

$l = 5, d = 15$ A1

[3 marks]

(c) finding the area of a square and the area of a (non-square) rectangle 5×5 and 5×15 (M1)

$(5 \times 5 + 5 \times 15 + 5 \times 15) \times 2$ (A1)

$= 350 \text{cm}^2$ A1

Note: The final answer is 350cm^2 . Units are required.

[3 marks]

(d) $\pi r^2 h = 375$ (A1)

$h = \frac{375}{\pi r^2}$ **A1**

Note: Award at most **A0A1** for a final answer $h = \frac{V}{\pi r^2}$.

[2 marks]

(e) attempt to substitute their h into the curved surface area $2\pi r h$ M1

$A = 2\pi r^2 + 2\pi r \left(\frac{375}{\pi r^2} \right)$ A1

$A = 2\pi r^2 + \frac{750}{r}$ AG

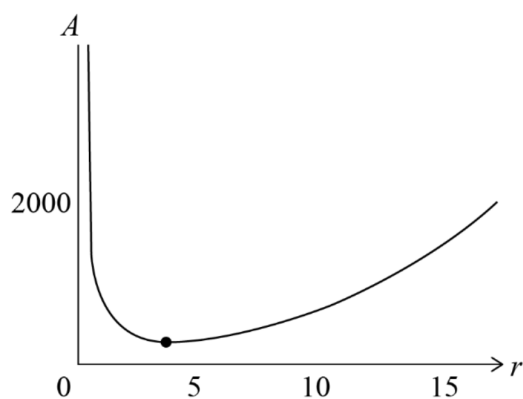
[2 marks]

(f) $\left(\frac{dA}{dr} \right) 4\pi r - 750r^{-2}$ A1A1A1

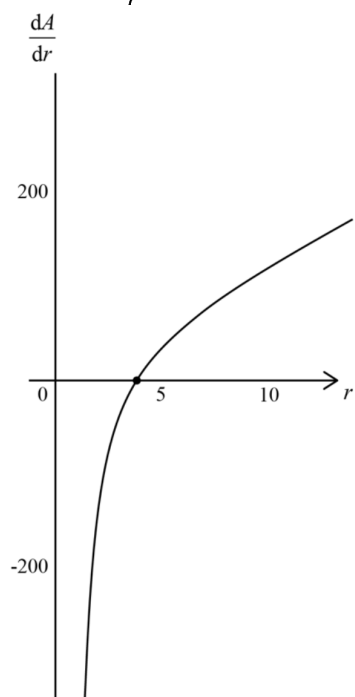
Note: Award **A1** for $4\pi r$, **A1** for -750 , **A1** for r^{-2} or equivalent. Award at most **A1A1A0** for additional terms seen.

[3 marks]

(g) (i)

METHOD 1correct sketch with some indication of min point (need not be labelled) **(M1)**

$$(r =) 3.91(\text{cm})(3.90796\dots)$$

A1**METHOD 2**attempt to solve $4\pi r - \frac{750}{r^2} = 0$, graphically or otherwise **(M1)**

$$(r =) 3.91(\text{cm})(3.90796\dots)$$

A1

$$(ii) \quad 288(\text{cm}^2) (287.873\dots)$$

A1**[3 marks]**

(h) (i) $\left(\frac{d^2A}{dr^2}\right) = 4\pi + \frac{1500}{r^3}$ **A1**

(ii) (as $r > 0$), $\frac{d^2A}{dr^2} > 0$ **R1**

hence graph is concave-up. **A1**

Note: Do not award **R0A1**. Marks can only be awarded in part (ii) if an expression is seen in part (i).

[3 marks]

Total [20 marks]

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