

IB Diploma

MATHEMATICS AA

HL P2
2017 — 2025

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1 - (MAT/22_HL_Summer_2017_Q11) - Functions - Roots, Graphs

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where a and b are positive integers.

- (a) Given that $x^2 - 1$ is a factor of $f(x)$ find the value of a and the value of b . [4]
- (b) Factorize $f(x)$ into a product of linear factors. [3]
- (c) Sketch the graph of $y = f(x)$, labelling the maximum and minimum points and the x and y intercepts. [3]
- (d) Using your graph state the range of values of c for which $f(x) = c$ has exactly two distinct real roots. [3]

2 - (MAT/21_HL_Summer_2018_Q2) - Functions - Roots

[Maximum mark: 6]

The equation $x^2 - 5x - 7 = 0$ has roots α and β . The equation $x^2 + px + q = 0$ has roots $\alpha + 1$ and $\beta + 1$. Find the value of p and the value of q .

3 - (MAT/22_HL_Summer_2018_Q10) - Functions - Roots

Consider the expression $f(x) = \tan\left(x + \frac{\pi}{4}\right)\cot\left(\frac{\pi}{4} - x\right)$.

- (a) (i) Sketch the graph of $y = f(x)$ for $-\frac{5\pi}{8} \leq x \leq \frac{\pi}{8}$.
- (ii) With reference to your graph, explain why f is a function on the given domain.
- (iii) Explain why f has no inverse on the given domain.
- (iv) Explain why f is not a function for $-\frac{3\pi}{4} \leq x \leq \frac{\pi}{4}$. [5]

The expression $f(x)$ can be written as $g(t)$ where $t = \tan x$.

- (b) Show that $g(t) = \left(\frac{1+t}{1-t}\right)^2$. [3]
- (c) Sketch the graph of $y = g(t)$ for $t \leq 0$. Give the coordinates of any intercepts and the equations of any asymptotes. [3]
- (d) Let α, β be the roots of $g(t) = k$, where $0 < k < 1$.
- (i) Find α and β in terms of k .
- (ii) Show that $\alpha + \beta < -2$. [7]

4 - (MathAA/22_HL_Summer_2021_Q4) - Algebra-functions - Roots

[Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

- (a) Find the range of f . [2]
- (b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c . [4]

5 - (MathAA/21_HL_Summer_2021_Q11) - Algebra-functions - Roots, Graphs

[Maximum mark: 20]

The function f is defined by $f(x) = \frac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

- (a) Find the value of p and the value of q . [2]
- (b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

- (c) Find the x -coordinate of the point of inflexion. [2]
- (d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \frac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\frac{2}{3}$.

- (e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]
- (f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

6 - (MathAA/22_HL_Summer_2021_Q12) - Algebra-functions - Roots, Differentiation

[Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}(x^2+1)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1} . [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

7 - (MathAA/20_HL_Winter_2021_Q10) - Algebra-functions - Roots, Graphs, Integration

[Maximum mark: 18]

Consider the function $f(x) = \frac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \frac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x -axis;

(ii) y -axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f . [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Q}$.

Find the value of a and the value of b . [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\frac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\int_0^3 \frac{1}{f(x)} dx$, expressing your answer as a single logarithm. [7]

8 - (MathAA/21_HL_Summer_2022_Q6) - Algebra-functions - Roots

[Maximum mark: 6]

Consider the function $f(x) = 2^x - \frac{1}{2^x}$, $x \in \mathbb{R}$.

- (a) Show that f is an odd function. [2]

The function g is given by $g(x) = \frac{x-1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

- (b) Solve the inequality $f(x) \geq g(x)$. [4]

9 - (MathAA/21_HL_Summer_2022_Q10) - Algebra-functions - Roots, Differentiation, Integration

[Maximum mark: 20]

Consider the function $f(x) = \sqrt{x^2-1}$, where $1 \leq x \leq 2$.

- (a) Sketch the curve $y = f(x)$, clearly indicating the coordinates of the endpoints. [2]

- (b) (i) Show that the inverse function of f is given by $f^{-1}(x) = \sqrt{x^2+1}$.

- (ii) State the domain and range of f^{-1} . [5]

The curve $y = f(x)$ is rotated 2π about the y -axis to form a solid of revolution that is used to model a water container.

- (c) (i) Show that the volume, $V \text{ m}^3$, of water in the container when it is filled to a height of h metres is given by $V = \pi \left(\frac{1}{3}h^3 + h \right)$.

- (ii) Hence, determine the maximum volume of the container. [5]

At $t = 0$, the container is empty. Water is then added to the container at a constant rate of $0.4 \text{ m}^3 \text{ s}^{-1}$.

- (d) Find the time it takes to fill the container to its maximum volume. [2]

- (e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

10 - (MathAA/22_HL_Summer_2022_Q7) - Algebra-functions - Roots

[Maximum mark: 8]

Consider $\lim_{x \rightarrow 0} \frac{\arctan(\cos x) - k}{x^2}$, where $k \in \mathbb{R}$.

- (a) Show that a finite limit only exists for $k = \frac{\pi}{4}$. [2]

- (b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

11 - (MathAA/20_HL_Winter_2022_Q11) - Algebra-functions - Roots, Differentiation

[Maximum mark: 15]

The function f is defined by $f(x) = e^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x -axis and the y -axis is rotated through 2π radians about the x -axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

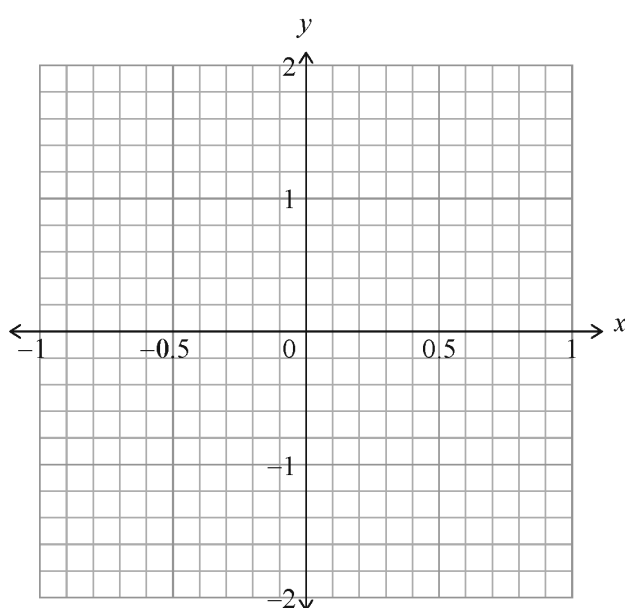
12 - (MathAA/21_HL_Summer_2023_Q2) - Algebra-functions - Roots

[Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \leq x \leq 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



13 - (MathAA/21_HL_Summer_2023_Q9) - Algebra-functions - Roots

[Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.**14** - (MathAA/22_HL_Summer_2023_Q8) - Algebra-functions - Roots

[Maximum mark: 7]

A function g is defined by $g(x) = \frac{2x-5}{x^2-3}$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$.

- (a) Determine the range of
- g
- . [4]

A function h is defined by $h(x) = g(|x|)\cos t$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$ and t is a constant where $\frac{\pi}{2} < t \leq \pi$.

- (b) Find the set of values of
- x
- such that
- $h(x) \leq 0$
- . [3]

15 - (MathAA/21_HL_Winter_2023_Q11) - Algebra-functions - Roots

[Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2-14x+24}{2x+6}$, where $x \in \mathbb{R}$, $x \neq -3$.

- (a) State the equation of the vertical asymptote on the graph of
- f
- . [1]

- (b) Find the coordinates of the points where the graph of
- f
- crosses the
- x
- axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

- (c) Find the value of
- a
- and the value of
- b
- . [4]

- (d) Sketch the graph of
- f
- for
- $-50 \leq x \leq 50$
- , showing clearly the asymptotes and any intersections with the axes. [4]

- (e) Find the range of
- f
- . [4]

- (f) Solve the inequality
- $f(x) > x$
- . [4]

16 - (MathAA/22_HL_Winter_2023_Q11) - Algebra-functions - Roots

[Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f . [1]

(b) Find the coordinates of the points where the graph of f crosses the x -axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b . [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f . [4]

(f) Solve the inequality $f(x) > x$. [4]

17 - (MathAA/21_HL_Summer_2024_Q1) - Algebra-functions - Roots

[Maximum mark: 7]

Darren buys a car for \$35 000. The value of the car decreases by 15% in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by 11% in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for n complete years, the value of the car is less than 10% of its original value.

(c) Find the least value of n . [3]

18 - (MathAA/2_HL_Winter_2024_Q1) - Algebra-functions - Roots

[Maximum mark: 7]

Consider the function $f(x) = 11\sqrt{x} - 2x - 11$, where $0 \leq x \leq 20$.

(a) Find the value of

(i) $f(0)$;

(ii) $f(20)$.

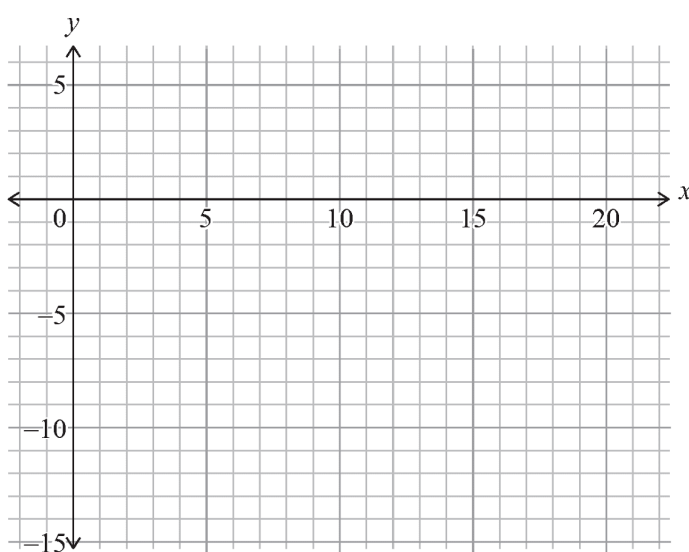
[2]

(b) Find the two roots of $f(x) = 0$.

[2]

(c) Sketch the graph of $y = f(x)$ on the following grid.

[3]



19 - (MathAA/2_HL_Winter_2024_Q12) - Algebra-functions - Roots, Differentiation

[Maximum mark: 21]

A curve C has equation $y = \frac{2x^2 + 6x - 3}{x + k}$, $x \in \mathbb{R}$, $x \neq -k$, where k is a real positive constant.

(a) Show that $\frac{dy}{dx} = \frac{2x^2 + 4kx + 6k + 3}{(x + k)^2}$. [4]

(b) Find the range of values of k for which a local minimum or maximum point exists. [4]

Consider the curve C , when $k = 2$.

(c) Write down the equation of the vertical asymptote. [1]

(d) Find the equation of the oblique asymptote. [4]

(e) Show that $\frac{dy}{dx} > 2$, for $x \in \mathbb{R}$, $x \neq -2$. [4]

(f) Sketch the curve C , showing clearly both asymptotes and the general behaviour of C as it approaches each asymptote. [You are not required to find any axes intercepts.] [4]

20 - (MathAA/21_HL_Summer_2025_Q2) - Trigonometry, Algebra-functions - Roots, Graphs

[Maximum mark: 4]

Consider the function $f(x) = a \tan(2x) + b$, where $x \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$ and $a, b \in \mathbb{R}$.

(a) Write down the period of f . [1]

The graph of $y = f(x)$ passes through the points $\left(\frac{\pi}{12}, 5\right)$ and $\left(\frac{\pi}{3}, 7\right)$.

(b) Find the value of a and the value of b . [3]

21 - (MathAA/21_HL_Summer_2025_Q6) - Trigonometry, Algebra-functions - Roots, Graphs

[Maximum mark: 7]

Consider the function $f(x) = 4 \cot x + \sin x$, where $0 < x < \pi$.

(a) (i) Write $f(x)$ in terms of $\sin x$ and $\cos x$.

(ii) Hence or otherwise, sketch the graph of $y = f(x)$, showing the value of the x -intercept. [3]

(b) Find the value of $f^{-1}(2)$. [1]

It is given that $\sec \alpha = 1.5$, where $0 < \alpha < \pi$.

(c) Find the value of $f(\alpha)$. [3]

22 - (MathAA/21_HL_Summer_2025_Q12) - Sequences - Series, Algebra-functions - Roots, Integration

[Maximum mark: 20]

Consider the family of functions f_n defined by $f_n(x) = \sum_{r=0}^n (-2x^2)^r$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that f_n is an even function for all values of n . [3]

(b) (i) Show that $f_3(x) = 1 - 2x^2 + 4x^4 - 8x^6$.

(ii) Write down a similar expression for $f_4(x)$ in ascending powers of x . [2]

Consider the function $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ defined over the domain $-k < x < k$ where $k > 0$.

The largest possible value of k is K .

(c) (i) Find the value of K , giving your answer in exact form.

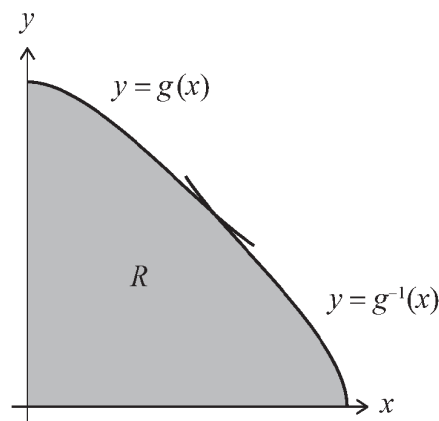
(ii) Express $f(x)$ as a rational function in the form $\frac{1}{a+bx^2}$, where a and b are constants to be determined. [5]

The function g is defined as $g(x) = f(x)$ for $0 \leq x < K$.

(d) (i) Justify that g^{-1} exists.

(ii) Find $g^{-1}(x)$, giving its domain. [6]

The region R is completely enclosed by the curves $y = g(x)$, $y = g^{-1}(x)$ and the x - and y -axes, as shown on the following diagram.



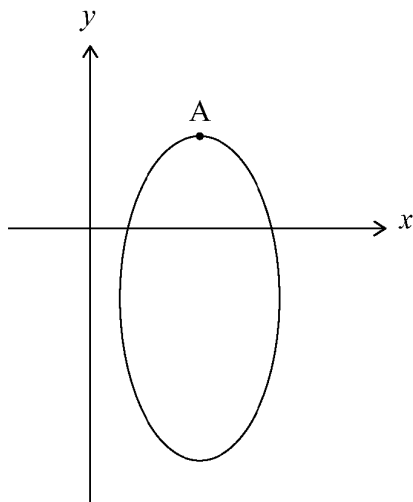
(e) Find the area of R . [4]

23 - (MathAA/22_HL_Summer_2025_Q12) - Differentiation, Algebra-functions - Roots, Integration

[Maximum mark: 22]

The curve C has equation $4x^2 + y^2 - 24x + 4y + 20 = 0$.

The following diagram shows C with a maximum point at A .



(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4(3-x)}{y+2}$. [4]

(b) Hence, determine the domain of C . Give your answer in the form $3 - \sqrt{a} \leq x \leq 3 + \sqrt{a}$, where $a \in \mathbb{Z}^+$. [4]

(c) Find (x_A, y_A) , the coordinates of A . [3]

A line $y = mx$ is a tangent to C , where $m \in \mathbb{Z}$.

(d) Find the possible values of m . [4]

The line $y = -4x$ touches C at point B .

(e) Find y_B , the y -coordinate of B . [3]

The region bounded by the curve C , the y -axis and the lines $y = y_A$ and $y = y_B$, is rotated 360° about the y -axis to form a solid of revolution.

(f) Find the volume of the solid formed. [4]

24 - (MathAA/23_HL_Winter_2025_Q6) - *Algebra-functions - Roots, Quadratics, Remainder & Factor Theorem*

[Maximum mark: 7]

Consider the quadratic polynomial given by $P(x) = 2x^2 + qx + r$ where $q, r \in \mathbb{R}$.

The equation $P(x) = 0$ has roots α and β , where $\alpha, \beta \in \mathbb{R}$.

When $P(x)$ is divided by $(x + 1)$, the remainder is 12.

Given that $\alpha^2 + \beta^2 = \frac{37}{4}$, find the value of q and the value of r .

25 - (MathAA/23_HL_Winter_2025_Q9) - *Algebra-functions - Roots*

[Maximum mark: 5]

Prove by contradiction that $\sqrt{2xy} \leq \frac{1}{2}(2x + y)$ for all $x, y \in \mathbb{R}^+$.

ANSWERS

1 - (MAT/22_HL_Summer_2017_Q11) - Functions - Roots, Graphs

(a) $g(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$

$$g(1) = 0 \Rightarrow a + b = 8$$

$$g(-1) = 0 \Rightarrow -a + b = -6$$

$$\Rightarrow a = 7, b = 1$$

M1A1

A1

A1

[4 marks]

(b) $3x^4 + 7x^3 + x^2 - 7x - 4 = (x^2 - 1)(px^2 + qx + r)$

attempt to equate coefficients

$$p = 3, q = 7, r = 4$$

$$3x^4 + 7x^3 + x^2 - 7x - 4 = (x^2 - 1)(3x^2 + 7x + 4)$$

$$= (x - 1)(x + 1)^2(3x + 4)$$

(M1)

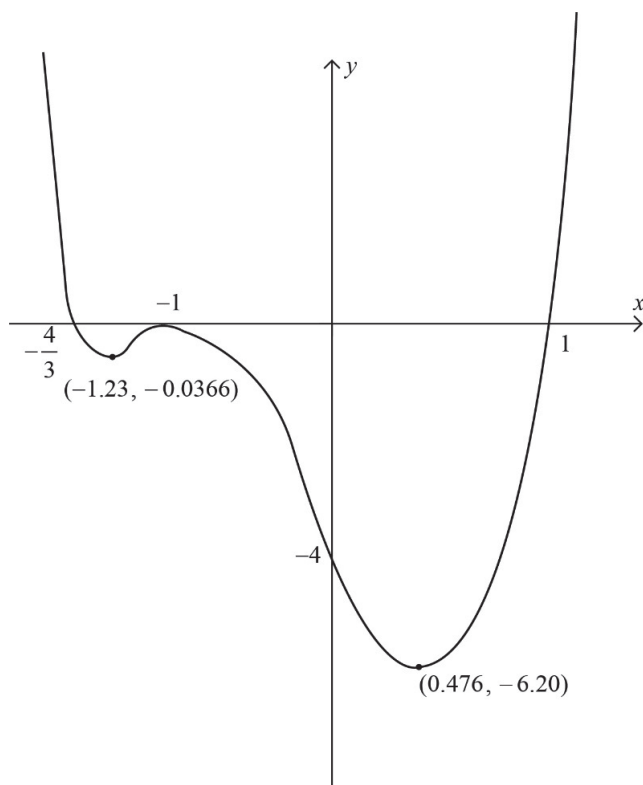
(A1)

A1

Note: Accept any equivalent valid method.

[3 marks]

(c)

**A1** for correct shape (ie with correct number of max/min points)**A1** for correct x and y intercepts**A1** for correct maximum and minimum points**[3 marks]**(d) $c > 0$ $-6.20 < c < -0.0366$ **A1****A1A1****Note:** Award **A1** for correct end points and **A1** for correct inequalities.**Note:** If the candidate has misdrawn the graph and omitted the first minimum point, the maximum mark that may be awarded is **A1FTA0A0** for $c > -6.20$ seen.**[3 marks]**

2 - (MAT/21_HL_Summer_2018_Q2) - Functions - Roots

METHOD 1

$$\alpha + \beta = 5, \alpha\beta = -7$$

(M1)(A1)

Note: Award **M1A0** if only one equation obtained.

$$(\alpha + 1) + (\beta + 1) = 5 + 2 = 7$$

A1

$$(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$$

(M1)

$$= -7 + 5 + 1 = -1$$

$$p = -7, q = -1$$

A1A1**METHOD 2**

$$\alpha = \frac{5 + \sqrt{53}}{2} = 6.1\dots; \beta = \frac{5 - \sqrt{53}}{2} = -1.1\dots$$

(M1)(A1)

$$\alpha + 1 = \frac{7 + \sqrt{53}}{2} = 7.1\dots; \beta + 1 = \frac{7 - \sqrt{53}}{2} = -0.1\dots$$

A1

$$(x - 7.14\dots)(x + 0.14\dots) = x^2 - 7x - 1$$

(M1)

$$p = -7, q = -1$$

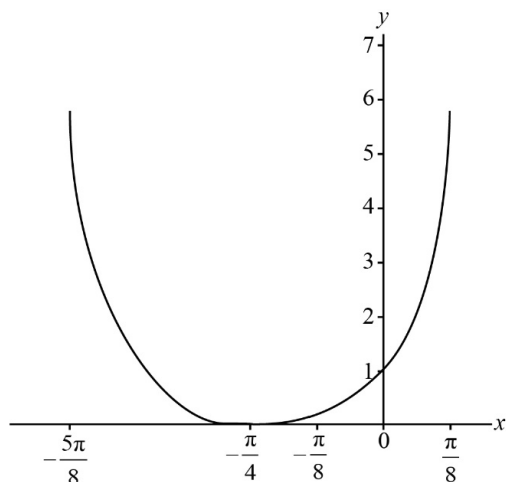
A1A1

Note: Exact answers only.

[6 marks]

3 - (MAT/22_HL_Summer_2018_Q10) - Functions - Roots

(a) (i)

**A1A1**

A1 for correct concavity, many to one graph, symmetrical about the midpoint of the domain and with two axes intercepts.

Note: Axes intercepts and scales not required.

A1 for correct domain

(ii) for each value of x there is a unique value of $f(x)$

A1

Note: Accept "passes the vertical line test" or equivalent.

(iii) no inverse because the function fails the horizontal line test or equivalent

R1

Note: No FT if the graph is in degrees (one-to-one).

(iv) the expression is not valid at either of $x = \frac{\pi}{4}$ (or $-\frac{3\pi}{4}$)

R1**[5 marks]**

(b) **METHOD 1**

$$f(x) = \frac{\tan\left(x + \frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$

M1

$$= \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}}$$

M1A1

$$= \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}$$

AG**METHOD 2**

$$f(x) = \tan\left(x + \frac{\pi}{4}\right) \tan\left(\frac{\pi}{2} - \frac{\pi}{4} + x\right)$$

(M1)

$$= \tan^2\left(x + \frac{\pi}{4}\right)$$

A1

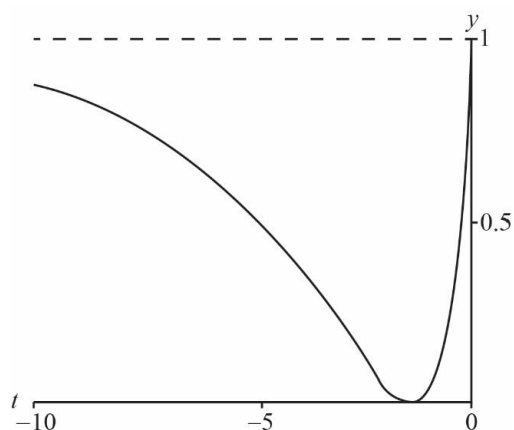
$$g(t) = \left(\frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} \right)^2$$

A1

$$= \left(\frac{1+t}{1-t} \right)^2$$

AG**[3 marks]**

(c)



for $t \leq 0$, correct concavity with two axes intercepts and with asymptote $y = 1$ **A1**

t intercept at $(-1, 0)$ **A1**

y intercept at $(0, 1)$ **A1**

[3 marks]

(d) (i) **METHOD 1**

$$\alpha, \beta \text{ satisfy } \frac{(1+t)^2}{(1-t)^2} = k \quad \mathbf{M1}$$

$$1 + t^2 + 2t = k(1 + t^2 - 2t) \quad \mathbf{A1}$$

$$(k-1)t^2 - 2(k+1)t + (k-1) = 0 \quad \mathbf{A1}$$

attempt at using quadratic formula **M1**

$$\alpha, \beta = \frac{k+1 \pm 2\sqrt{k}}{k-1} \text{ or equivalent} \quad \mathbf{A1}$$

METHOD 2

$$\alpha, \beta \text{ satisfy } \frac{1+t}{1-t} = (\pm)\sqrt{k} \quad \mathbf{M1}$$

$$t + \sqrt{k}t = \sqrt{k} - 1 \quad \mathbf{M1}$$

$$t = \frac{\sqrt{k}-1}{\sqrt{k}+1} \text{ (or equivalent)} \quad \mathbf{A1}$$

$$t - \sqrt{k}t = -(\sqrt{k}+1) \quad \mathbf{M1}$$

$$t = \frac{\sqrt{k}+1}{\sqrt{k}-1} \text{ (or equivalent)} \quad \mathbf{A1}$$

$$\text{so for eg, } \alpha = \frac{\sqrt{k}-1}{\sqrt{k}+1}, \beta = \frac{\sqrt{k}+1}{\sqrt{k}-1}$$

$$(ii) \quad \alpha + \beta = 2 \frac{(k+1)}{(k-1)} \left(= -2 \frac{(1+k)}{(1-k)} \right)$$

$$\text{since } 1+k > 1-k$$

$$\alpha + \beta < -2$$

A1**R1****AG**

Note: Accept a valid graphical reasoning.

[7 marks]

4 - (MathAA/22_HL_Summer_2021_Q4) - Algebra-functions - Roots

(a) attempting to find the vertex

(M1)

$$x=1 \text{ OR } y=-5 \text{ OR } f(x)=6(x-1)^2-5$$

$$\text{range is } y \geq -5$$

A1**[2 marks]**

(b) **METHOD 1**

$$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \quad \left(= -(6(x-1)^2 - 5) + c \right)$$

(A1)**EITHER**

relating to the range of f OR attempting to find $g(-5)$

(M1)

$$5+c \leq 0$$

(A1)**OR**

attempting to find the discriminant of $(g \circ f)(x)$

(M1)

$$144 + 24(c-1) \leq 0 \quad (120 + 24c \leq 0)$$

(A1)**THEN**

$$c \leq -5$$

A1**[4 marks]****METHOD 2**

vertical reflection followed by vertical shift

(M1)

new vertex is $(1, 5+c)$

(A1)

$$5+c \leq 0$$

(A1)

$$c \leq -5$$

A1**[4 marks]**

5 - (MathAA/21_HL_Summer_2021_Q11) - Algebra-functions - Roots, Graphs

(a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ **(M1)**

$$p = \frac{1}{2}, q = -\frac{1}{2} \text{ or vice versa}$$

A1**[2 marks]**

(b) attempt to use quotient rule or product rule

(M1)**EITHER**

$$f'(x) = \frac{3(4x^2 - 1) - 8x(3x + 2)}{(4x^2 - 1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2 - 1)^2} \right)$$

A1A1

Note: Award **A1** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x + 2)(4x^2 - 1)^{-2} + 3(4x^2 - 1)^{-1}$$

A1A1

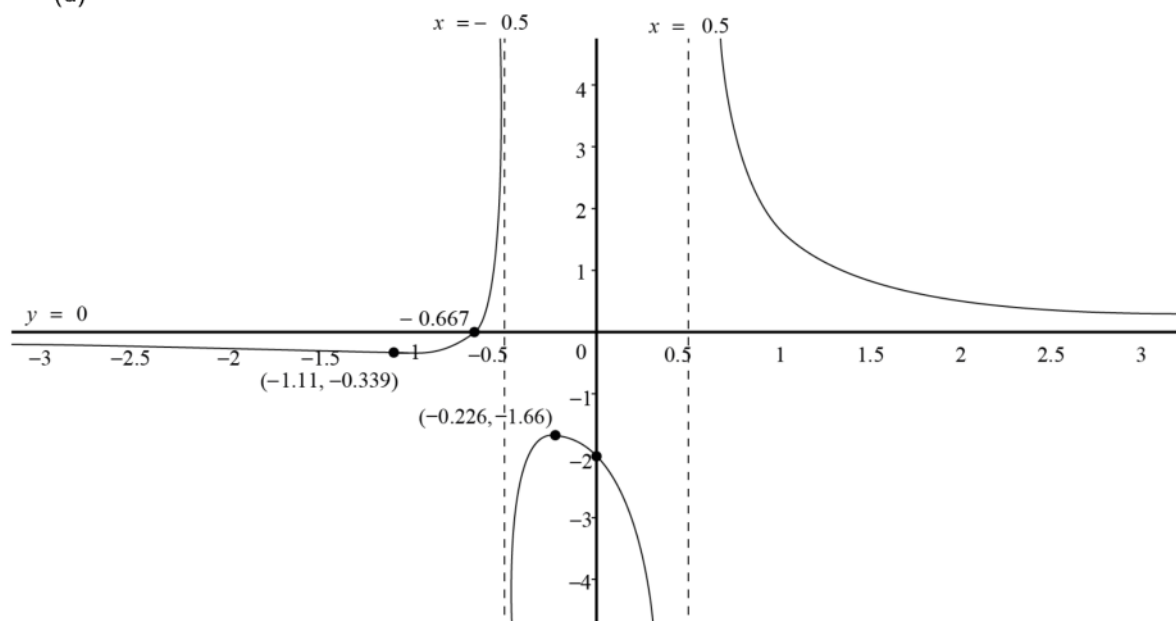
Note: Award **A1** for each term.

[3 marks](c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ **(M1)**

$$x = -1.60$$

A1**[2 marks]**

(d)

**A1A1A1A1A1**

Note: Award **A1** for both vertical asymptotes with their equations, award **A1** for horizontal asymptote with equation, award **A1** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum **A0A1A0A1A1**.

[5 marks]

e) $x = -\frac{2}{3} (= -0.667)$

A1

(oblique asymptote has) gradient $\frac{4}{3} (= 1.33)$

(A1)

appropriate method to find complete equation of oblique asymptote

M1

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x + 2 \overline{) 4x^2 + 0x - 1} \end{array}$$

$$4x^2 + \frac{8}{3}x$$

$$\hline -\frac{8}{3}x - 1$$

$$-\frac{8}{3}x - \frac{16}{9}$$

$$\hline \frac{7}{9}$$

$$y = \frac{4}{3}x - \frac{8}{9} (= 1.33x - 0.889)$$

A1

Note: Do not award the final **A1** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value ($x = -0.568729\dots, x = 1.31872\dots$) **(M1)**

$$-\frac{2}{3} < x < -0.569 \quad \text{OR} \quad -0.5 < x < 0.5 \quad \text{OR} \quad x > 1.32$$

A1A1A1

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]

6 - (MathAA/22_HL_Summer_2021_Q12) - Algebra-functions - Roots, Differentiation

(a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x) \quad \textbf{R1}$$

OR

a sketch graph of $y = f(x)$ with line symmetry in the y -axis indicated **R1**

THEN

so $f(x)$ is an even function **AG**

[1 mark]

(b) as $x \rightarrow \pm\infty$, $f(x) \rightarrow \arcsin 1 \left(\rightarrow \frac{\pi}{2} \right)$ **A1**

so the horizontal asymptote is $y = \frac{\pi}{2}$ **A1**

[2 marks]

(c) (i) attempting to use the quotient rule to find $\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$

M1

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \left(= \frac{4x}{(x^2+1)^2} \right)$$

A1

attempting to use the chain rule to find $\frac{d}{dx}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$

M1

let $u = \frac{x^2-1}{x^2+1}$ and so $y = \arcsin u$ and $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \times \frac{4x}{(x^2+1)^2}$$

M1

$$= \frac{4x}{\sqrt{(x^2+1)^2 - (x^2-1)^2}} \times \frac{1}{(x^2+1)}$$

A1

$$= \frac{4x}{\sqrt{4x^2}} \times \frac{1}{(x^2+1)}$$

A1

$$= \frac{2x}{\sqrt{x^2}(x^2+1)}$$

AG

(ii) $f'(x) = \frac{2x}{|x|(x^2+1)}$

EITHER

for $x < 0$, $|x| = -x$

(A1)

so $f'(x) = -\frac{2}{x^2+1}$

A1**OR**

$|x| > 0$ and $x^2 + 1 > 0$

A1

$2x < 0$, $x < 0$

A1**THEN**

$f'(x) < 0$

R1

Note: Award **R1** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x < 0$

AG

Note: Do not accept a graphical solution.

[9 marks]

(d) $x = \arcsin\left(\frac{y^2 - 1}{y^2 + 1}\right)$

M1

$$\sin x = \frac{y^2 - 1}{y^2 + 1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$$

A1

$$y^2 = \frac{1 + \sin x}{1 - \sin x}$$

A1

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected)

R1

Note: The **R1** is dependent on the above **A1**.

$$\text{so } (g^{-1}(x)) = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$$

A1

Note: The final **A1** is not dependent on **R1** mark.

[5 marks]

(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$

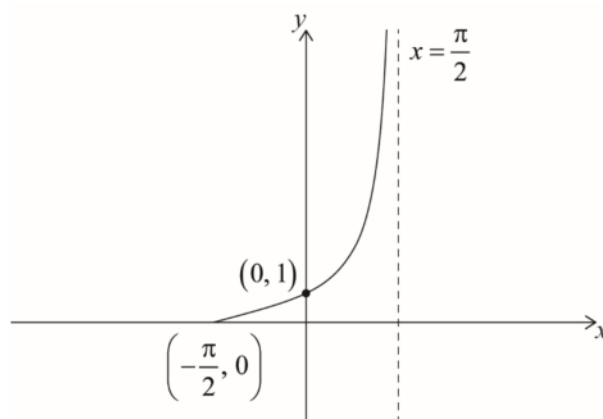
A1

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

(f)

**A1A1A1****Note:** **A1** for correct domain and correct range and y -intercept at $y = 1$ **A1** for asymptotic behaviour $x \rightarrow \frac{\pi}{2}$ **A1** for $x = \frac{\pi}{2}$

Coordinates are not required.

Do not accept $x = 1.57$ or other inexact values.**[3 marks]**

7 - (MathAA/20_HL_Winter_2021_Q10) - Algebra-functions - Roots, Graphs, Integration

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ **(M1)** $(-3, 0)$ and $(4, 0)$ **A1**(ii) $\left(0, \frac{4}{5}\right)$ **A1****[3 marks]**(b) $x = \frac{15}{2}$ **A1**

Note: Award **A0** for $x \neq \frac{15}{2}$.

Award **A1** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark](c) **METHOD 1**

$$(ax + b)(2x - 15) \equiv x^2 - x - 12$$

attempts to expand $(ax + b)(2x - 15)$ **(M1)**

$$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$$

$$a = \frac{1}{2}$$

A1equates coefficients of x **(M1)**

$$-1 = -\frac{15}{2} + 2b$$

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4}\right)$$

METHOD 2attempts division on $\frac{x^2 - x - 12}{2x - 15}$ **M1**

$$\frac{x}{2} + \frac{13}{4} + \dots$$

M1

$$a = \frac{1}{2}$$

A1

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

METHOD 3

$$a = \frac{1}{2}$$

A1

$$\frac{x^2 - x - 12}{2x - 15} \equiv \frac{x}{2} + b + \frac{c}{2x - 15}$$

M1

$$x^2 - x - 12 \equiv \frac{(2x - 15)x}{2} + (2x - 15)b + c$$

equates coefficients of x :**(M1)**

$$-1 = -\frac{15}{2} + 2b$$

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

METHOD 4attempts division on $\frac{x^2 - x - 12}{2x - 15}$ **M1**

$$\frac{x^2 - x - 12}{2x - 15} = \frac{x}{2} + \frac{\frac{13x}{2} - 12}{2x - 15}$$

$$a = \frac{1}{2}$$

A1

$$\frac{\frac{13x}{2} - 12}{2x - 15} = \frac{13}{4} + \dots$$

M1

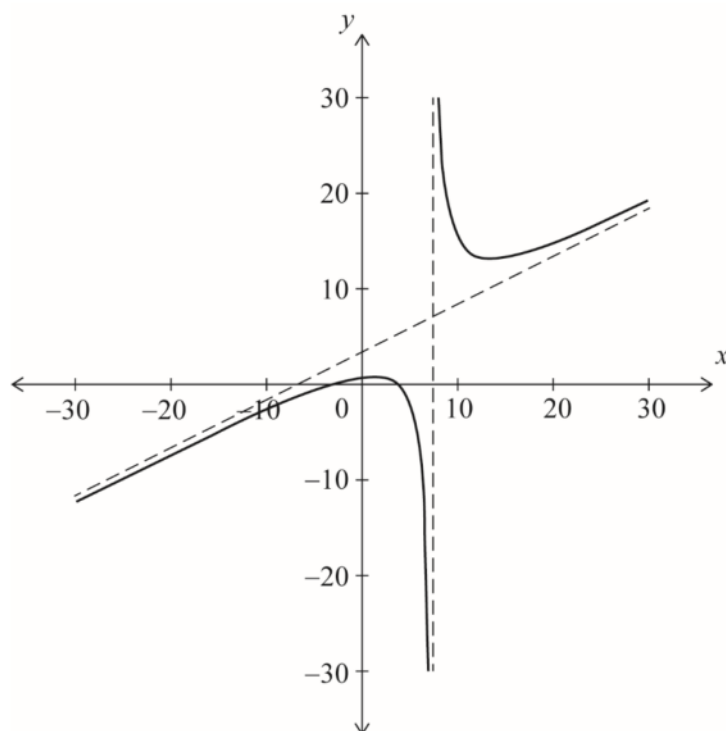
$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \leq x \leq 30$)

A1

their vertical and oblique asymptotes in approximately correct positions with both branches

showing correct asymptotic behaviour to these asymptotes

A1

their axes intercepts in approximately the correct positions

A1

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions:

(M1)

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4) + B(x+3)$$

$$A = 3$$

A1

$$B = -1$$

A1

$$\left(\frac{3}{x+3} - \frac{1}{x-4} \right)$$

$$(ii) \int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4} \right) dx$$

attempts to integrate and obtains two terms involving 'ln'

(M1)

$$= \left[3 \ln|x+3| - \ln|x-4| \right]_0^3$$

A1

$$= 3 \ln 6 - \ln 1 - 3 \ln 3 + \ln 4$$

A1

$$= 3 \ln 2 + \ln 4 \quad (= \ln 8 + \ln 4)$$

$$= \ln 32 \quad (= 5 \ln 2)$$

A1

Note: The final A1 is dependent on the previous two A marks.

[7 marks]

Total [18 marks]

8 - (MathAA/21_HL_Summer_2022_Q6) - Algebra-functions - Roots

- (a) attempt to replace
- x
- with
- $-x$

M1

$$f(-x) = 2^{-x} - \frac{1}{2^{-x}}$$

EITHER

$$= \frac{1}{2^x} - 2^x = -f(x)$$

A1**OR**

$$= -\left(2^x - \frac{1}{2^x}\right) (= -f(x))$$

A1

Note: Award **M1A0** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y -axis and then in the x -axis (or vice versa).

so f is an odd function**AG****[2 marks]**

- (b) attempt to find at least one intersection point

(M1)

$$x = -1.26686..., x = 0.177935..., x = 3.06167...$$

$$x = -1.27, x = 0.178, x = 3.06$$

$$-1.27 \leq x < -1,$$

A1

$$0.178 \leq x < 3,$$

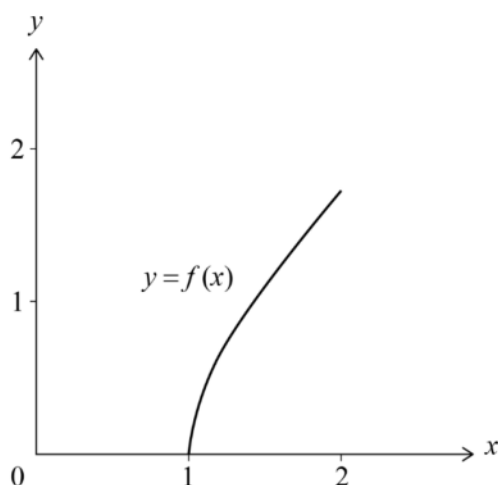
A1

$$x \geq 3.06$$

A1**[4 marks]**

9 - (MathAA/21_HL_Summer_2022_Q10) - Algebra-functions - Roots, Differentiation, Integration

(a)



correct shape (concave down) within the given domain $1 \leq x \leq 2$

A1

$(1, 0)$ and $(2, \sqrt{3}) (= (2, 1.73))$

A1

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

(b) (i) interchanging x and y (seen anywhere)

M1

$$x = \sqrt{y^2 - 1}$$

$$x^2 = y^2 - 1$$

A1

$$y = \sqrt{x^2 + 1}$$

A1

$$f^{-1}(x) = \sqrt{x^2 + 1}$$

AG

(ii) $0 \leq x \leq \sqrt{3}$ OR domain $[0, \sqrt{3}] (= [0, 1.73])$

A1

$1 \leq y \leq 2$ OR $1 \leq f^{-1}(x) \leq 2$ OR range $[1, 2]$

A1

[5 marks]

- (c) (i) attempt to substitute $x = \sqrt{y^2 + 1}$ into the correct volume formula (M1)

$$V = \pi \int_0^h \left(\sqrt{y^2 + 1} \right)^2 dy \left(= \pi \int_0^h (y^2 + 1) dy \right) \quad \text{A1}$$

$$= \pi \left[\frac{1}{3} y^3 + y \right]_0^h \quad \text{A1}$$

$$= \pi \left(\frac{1}{3} h^3 + h \right) \quad \text{AG}$$

Note: Award marks as appropriate for correct work using a different variable e.g.

$$\pi \int_0^h \left(\sqrt{x^2 + 1} \right)^2 dx$$

- (ii) attempt to substitute $h = \sqrt{3}$ ($= 1.732\dots$) into V (M1)

$$V = 10.8828\dots$$

$$V = 10.9 \text{ (m}^3\text{)} \left(= 2\sqrt{3}\pi \right) \text{ (m}^3\text{)} \quad \text{A1}$$

[5 marks]

- (d) **METHOD 1**

$$\text{time} = \frac{10.8828\dots}{0.4} \left(= \frac{2\sqrt{3}\pi}{0.4} \right) \quad \text{(M1)}$$

$$= 27.207\dots$$

$$= 27.2 \left(= 5\sqrt{3}\pi \right) \text{ (s)} \quad \text{A1}$$

[2 marks]

(e) attempt to find the height of the tank when $V = 5.4414... (= \sqrt{3}\pi)$ **(M1)**

$$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414... (= \sqrt{3}\pi)$$

$$h = 1.1818... \quad \textbf{(A1)}$$

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t **(M1)**

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\pi(h^2 + 1)} \times \frac{dV}{dt} \text{ OR } \frac{dV}{dt} = \pi(h^2 + 1) \frac{dh}{dt} \quad \textbf{(A1)}$$

attempt to substitute **their** h and $\frac{dV}{dt} = 0.4$ **(M1)**

$$\frac{dh}{dt} = \frac{0.4}{\pi(1.1818...^2 + 1)} = 0.053124...$$

$$= 0.0531 \text{ (m s}^{-1}\text{)}$$

A1**[6 marks]**

10 - (MathAA/22_HL_Summer_2022_Q7) - Algebra-functions - Roots

(a) (as $\lim_{x \rightarrow 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$$\Rightarrow \lim_{x \rightarrow 0} (\arctan(\cos x) - k) = 0 \quad \textbf{M1}$$

$$\arctan 1 - k = 0 \quad (k = \arctan 1) \quad \textbf{A1}$$

$$\text{so } k = \frac{\pi}{4} \quad \textbf{AG}$$

Note: Award **M1A0** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

$$(b) \lim_{x \rightarrow 0} \frac{\arctan(\cos x) - \frac{\pi}{4}}{x^2} \left(= \frac{0}{0} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\frac{-\sin x}{1 + \cos^2 x}}{2x}$$

A1A1

Note: Award **A1** for a correct numerator and **A1** for a correct denominator.

recognises to apply l'Hôpital's rule again

(M1)

$$= \lim_{x \rightarrow 0} \frac{\frac{-\sin x}{1 + \cos^2 x}}{2x} \left(= \frac{0}{0} \right)$$

Note: Award **M0** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$$= \lim_{x \rightarrow 0} \frac{\frac{-\cos x(1 + \cos^2 x) - 2\sin^2 x \cos x}{(1 + \cos^2 x)^2}}{2}$$

A1A1

Note: Award **A1** for a correct first term in the numerator and **A1** for a correct second term in the numerator.

OR

$$\lim_{x \rightarrow 0} \frac{-\cos x}{2(1 + \cos^2 x) - 4x \sin x \cos x}$$

A1A1

Note: Award **A1** for a correct numerator and **A1** for a correct denominator.

THEN

substitutes $x = 0$ into the correct expression to evaluate the limit

A1

Note: The final **A1** is dependent on all previous marks.

$$= -\frac{1}{4}$$

AG**[6 marks]**

11 - (MathAA/20_HL_Winter_2022_Q11) - Algebra-functions - Roots, Differentiation

(a) attempt to use product rule (M1)

$$f'(x) = 3e^{2x} + 2e^{2x}(3x - 4) (= e^{2x}(6x - 5)) \quad \text{A2}$$

Note: Award **A1** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ (M1)

$$x = 0.86299\dots$$

$$x = 0.863 \quad \text{A1}$$

$$y = -7.92719\dots$$

$$y = -7.93 \quad \text{A1}$$

$$(0.863, -7.93)$$

[3 marks]

(c) x -intercept is at $\frac{4}{3}(1.33)$ (A1)

attempt to use formula for volume of revolution (M1)

Note: Award (M1) for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$$\pi \int_0^{\frac{4}{3}} (e^{2x}(3x - 4))^2 dx \quad \text{(A1)}$$

Note: This (A1) can be awarded if the dx is omitted.

$$= 164.849\dots$$

$$= 165 \quad \text{A1}$$

[4 marks]

- (d) (i) attempt to compose functions in the correct order

(M1)

$$(f \circ g)(0) = f(g(0)) = f(1)$$

$$= -7.38905\dots$$

$$= -7.39 (= -e^2)$$

A1

- (ii) attempt to use the chain rule

(M1)

$$(f \circ g)'(0) = f'(g(0))g'(0)$$

Note: For this **(M1)** to be awarded, multiplication of two derivatives should be seen or implied.

$$= 2f'(1) (= 2 \times 7.38905\dots)$$

(A1)

$$= 14.7781\dots$$

$$= 14.8 (= 2e^2)$$

A1

[5 marks]

12 - (MathAA/21_HL_Summer_2023_Q2) - Algebra-functions - Roots

(a) attempt to substitute g into f

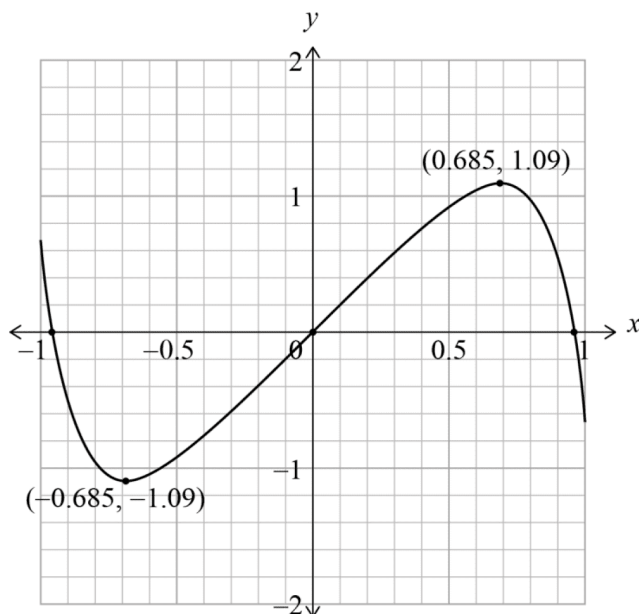
(M1)

$$(f \circ g)(x) = 2 \tan x - \tan^3 x$$

A1

[2 marks]

(b)



A1A1A1

Note: A1 for approximately correct odd function passing through the origin with a maximum above $y = 1$ and a minimum below $y = -1$.

A1 for endpoints at $x = \pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled

$(0.685, 1.09)$ AND minimum in approximately correct position and labelled

$(-0.685, -1.09)$. For approximate position, allow $-0.8 \leq x \leq -0.6$,

$-1.2 \leq y \leq -1$ for minimum and $0.6 \leq x \leq 0.8$, $1 \leq y \leq 1.2$ for maximum. If

the candidate gives the coordinates of extrema below their sketch, only

award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

13 - (MathAA/21_HL_Summer_2023_Q9) - Algebra-functions - Roots

Assume $p^2 - 8q - 11 = 0$, $(p, q \in \mathbb{Z})$ **M1**

Note: This **M1** is dependent on the assumption of truth (implied by “assume” or “suppose that ... is true”).

Subsequent marks should be awarded independently.

EITHER

$$p^2 = 8q + 11 (= 2(4q + 5) + 1) \text{ so } p^2 \text{ odd} \Rightarrow p \text{ odd}$$

R1**OR**

$$p \text{ even} \Rightarrow p^2 - 8q = 11 \text{ even which is a contradiction so } p \text{ is odd}$$

R1

Note: This **R1** should be awarded for any valid reason to conclude that p must be odd.

THEN

$$p = 2k + 1, (k \in \mathbb{Z})$$

M1

$$(2k + 1)^2 = 8q + 11$$

$$4k^2 + 4k + 1 = 8q + 11$$

(A1)

$$4k^2 + 4k = 8q + 10$$

$$2k^2 + 2k = 4q + 5 \text{ or equivalent with one side odd and one side even}$$

A1

a contradiction as LHS is even and RHS is odd

R1

Note: This **R1** is dependent on all previous marks.

Accept correct variations such as work based on $p = 2k - 1$.

therefore, if $p, q \in \mathbb{Z}$ then $p^2 - 8q - 11 \neq 0$

AG**Total [6 marks]**

14 - (MathAA/22_HL_Summer_2023_Q8) - Algebra-functions - Roots

(a) EITHER

attempts to find the y -coordinate of either the local minimum point or the local maximum point

(M1)

OR

attempts to find the discriminant of $2x - 5 = y(x^2 - 3)$ ($yx^2 - 2x + (5 - 3y) = 0$)

(M1)

$$\Delta = 4 - 4y(5 - 3y) (= 4 - 20y + 12y^2)$$

THEN

 $y = 1.43425\dots$ (local min.) and $y = 0.232408\dots$ (local max.)

(A1)(A1)

$$g(x) \leq 0.232 \text{ OR } g(x) \geq 1.43 \quad (g(x) \leq \frac{-\sqrt{13}+5}{6} \text{ OR } g(x) \geq \frac{\sqrt{13}+5}{6})$$

A1

Note: Accept other valid notations such as interval notation.

[4 marks]

$$(b) \frac{2|x|-5}{x^2-3} \geq 0 \text{ (since } \cos t < 0 \text{ for } \frac{\pi}{2} < t \leq \pi)$$

(R1)

attempts to solve graphically or algebraically

(M1)

$$x \leq -\frac{5}{2} \text{ OR } -\sqrt{3} < x < \sqrt{3} \text{ } (-1.73 < x < 1.73) \text{ OR } x \geq \frac{5}{2}$$

A1

[3 marks]

15 - (MathAA/21_HL_Winter_2023_Q11) - Algebra-functions - Roots

(a) (vertical asymptote equation) $x = -3$ **A1**

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark](b) $(2, 0)$ and $(12, 0)$ **A1A1**

Note: Award **A1** for $(2, 0)$ and **A1** for $(12, 0)$.

Award **A1A0** if only x values are given.

[2 marks](c) **METHOD 1**

$$a = \frac{1}{2}$$

A1

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$

(M1)

$$\frac{x^2 - 14x + 24}{2x + 6}$$

$$= \frac{1}{2}x - \frac{17}{2} \left(+ \frac{\dots}{2x + 6} \right)$$

(A1)

$$b = -\frac{17}{2}$$

A1

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 2

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6} \quad \text{(A1)}$$

$$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$$

attempt to equate coefficients of x : (M1)

$$-14 = 3 + 2b$$

$$b = -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6} \quad \text{(A1)}$$

attempt to find the limit of $f(x) - ax$ as $x \rightarrow \infty$ (M1)

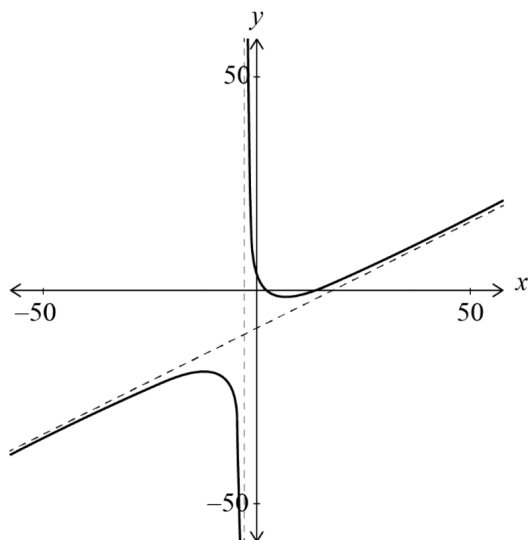
$$b = \lim_{x \rightarrow \infty} \frac{-17x + 24}{2x + 6}$$

$$= -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

(d)

two branches with approximately correct shape (for $-50 \leq x \leq 50$)**A1**

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes

A1A1

Note: Award **A1** for vertical asymptote and behaviour and **A1** for oblique asymptote and behaviour.

If only top half of the graph seen only award **A1A0** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions

A1

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $(-10 - 5\sqrt{3}) = -18.6602\dots$ OR $(-10 + 5\sqrt{3}) = -1.33974\dots$ seen anywhere **(A1)**

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$$y \leq -18.7, y \geq -1.34$$

A1A1

Note: Award **A1** for each inequality. Award **A1A0** for strict inequalities in both.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

(f) $(-10 - 2\sqrt{31}) = -21.1355\dots$ OR $(-10 + 2\sqrt{31}) = 1.13552\dots$ seen anywhere **(A1)**

$$x < -21.1, -3 < x < 1.14$$

A1A1A1

Note: Award **A1** for $x < -21.1$, **A1** for correct endpoints of a single interval -3 and 1.14 and **A1** for $-3 < x < 1.14$.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

16 - (MathAA/22_HL_Winter_2023_Q11) - Algebra-functions - Roots

(a) (vertical asymptote equation) $x = -3$ **A1****Note:** Accept $2x + 6 = 0$ or equivalent.**[1 mark]**(b) $(2, 0)$ and $(12, 0)$ **A1A1****Note:** Award **A1** for $(2, 0)$ and **A1** for $(12, 0)$.Award **A1A0** if only x values are given.**[2 marks]**(c) **METHOD 1**

$$a = \frac{1}{2}$$

A1attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ **(M1)**

$$\frac{x^2 - 14x + 24}{2x + 6}$$

$$= \frac{1}{2}x - \frac{17}{2} \left(+ \frac{\dots}{2x + 6} \right)$$

(A1)

$$b = -\frac{17}{2}$$

A1**Note:** Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 2

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6} \quad \text{(A1)}$$

$$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$$

attempt to equate coefficients of x : (M1)

$$-14 = 3 + 2b$$

$$b = -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6} \quad \text{(A1)}$$

attempt to find the limit of $f(x) - ax$ as $x \rightarrow \infty$ (M1)

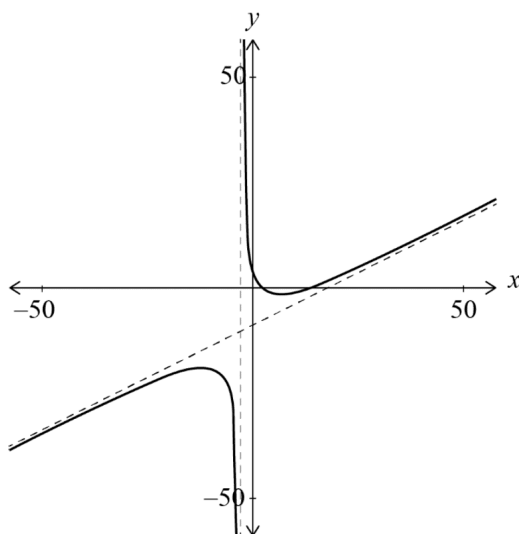
$$b = \lim_{x \rightarrow \infty} \frac{-17x + 24}{2x + 6}$$

$$= -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

(d)



two branches with approximately correct shape (for $-50 \leq x \leq 50$)

A1

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes

A1A1

Note: Award **A1** for vertical asymptote and behaviour and **A1** for oblique asymptote and behaviour.

If only top half of the graph seen only award **A1A0** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions

A1

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $(-10 - 5\sqrt{3}) = -18.6602\dots$ OR $(-10 + 5\sqrt{3}) = -1.33974\dots$ seen anywhere **(A1)**

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$$y \leq -18.7, y \geq -1.34$$

A1A1

Note: Award **A1** for each inequality. Award **A1A0** for strict inequalities in both.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

(f) $(-10 - 2\sqrt{31}) = -21.1355\dots$ OR $(-10 + 2\sqrt{31}) = 1.13552\dots$ seen anywhere **(A1)**

$$x < -21.1, -3 < x < 1.14$$

A1A1A1

Note: Award **A1** for $x < -21.1$, **A1** for correct endpoints of a single interval -3 and 1.14 and **A1** for

$-3 < x < 1.14$.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

17 - (MathAA/21_HL_Summer_2024_Q1) - Algebra-functions - Roots

- (a) recognition that a 15% loss leaves 85% OR finding 15% and subtracting from original **(M1)**

$$0.85 \times 35000 \text{ OR } 35000 - 0.15 \times 35000$$

$$= (\$)29750$$

A1

Note: Accept $(\$)29800$.

[2 marks]

- (b) **EITHER**

$$29750 \times 0.89^9$$

(A1)

OR

$$N = 9$$

$$I\% = -11$$

$$PV = \mp 29750$$

(A1)

THEN

$$\text{value}(FV) = (\$)10423$$

A1

Note: For this **A1** the answer must be rounded to the nearest dollar.

Accept $(\$)10441$ from using 3 sf answer from part (a).

[2 marks]

(c) **METHOD 1**

attempt to solve the inequality (or equation) $29750 \times 0.89^{n-1} < 3500$ OR table of values **(M1)**

19.3643... OR $(n = 19 \Rightarrow) 3651.80...$ OR $(n = 20 \Rightarrow) 3250.10...$ **(A1)**

Note: For candidates using (\$)29800, $n > 19.3787...$, $(n = 19 \Rightarrow) 3657.93...$,
 $(n = 20 \Rightarrow) 3255.56...$

$n = 20$

A1

[3 marks]

METHOD 2

use of the finance app with $I\% = -11$, $PV = \mp 29750$, $FV = \pm 3500$

OR $29750 \times 0.89^N < 3500$ (condone the use of n or x) **(M1)**

$(N =) 18.3643...$ **(A1)**

Note: For candidates using (\$)29800, $N = 18.3787...$

$n = 20$

A1

[3 marks]

Total [7 marks]

18 - (MathAA/2_HL_Winter_2024_Q1) - Algebra-functions - Roots

(a) (i) $f(0) = -11$ **A1**

(ii) $-1.80650...$

$f(20) = -1.81 \left(= 11\sqrt{20} - 51 = 22\sqrt{5} - 51 \right)$ **A1**

[2 marks]

(b) attempt to find at least one root **(M1)**

$x = 1.72622...$ and $x = 17.5237...$

$x = 1.73$ and $x = 17.5$

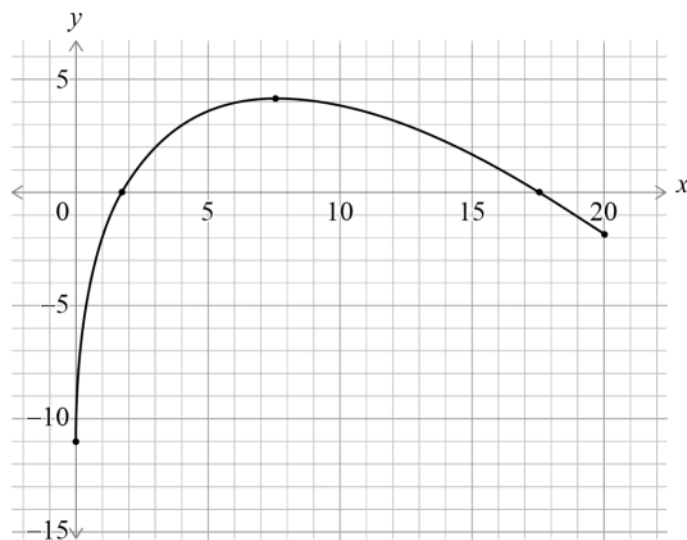
A1

[2 marks]

continued...

Question 1 continued.

(c)



A1A1A1

Note: Follow through from their part (a).

Award **A1** for endpoints at approximately $(0, -11)$ and $(20, -1.8)$. Allow for y -intercept $-12 < y < -10$ and for the right endpoint in the interval $x = 20, -2 < y < -1$.

The following two **A** marks may only be awarded if the approximate shape is correct.

Award **A1** for x -intercepts at approximately $x = 1.7$ and $x = 17.5$ and award **A1** for maximum at approximately $(7.6, 4.1)$. Allow for x -intercepts in the intervals $1 < x < 2$, $17 < x < 18$, and maximum in the intervals $6.5 < x < 8.5$, $3.5 < y < 5$.

[3 marks]

Total [7 marks]

19 - (MathAA/2_HL_Winter_2024_Q12) - Algebra-functions - Roots, Differentiation

(a) attempt to use quotient rule

M1

$$\frac{dy}{dx} = \frac{(x+k)(4x+6) - (2x^2+6x-3)}{(x+k)^2}$$

A1A1

Note: Award **A1** for $\frac{(x+k)(4x+6)}{(x+k)^2}$, **A1** for $\frac{-(2x^2+6x-3)}{(x+k)^2}$ OR $\frac{-2x^2-6x+3}{(x+k)^2}$

$$= \frac{4x^2 + 4kx + 6x + 6k - 2x^2 - 6x + 3}{(x+k)^2} \text{ or equivalent leading to AG line}$$

A1

$$= \frac{2x^2 + 4kx + 6k + 3}{(x+k)^2}$$

AG

Note: Candidates may use product rule, give marks accordingly.

[4 marks]

(b) local min or max when $\frac{dy}{dx} = 0$, when $2x^2 + 4kx + 6k + 3 = 0$ has real solutions.

attempt to find the discriminant of numerator

(M1)

$$(4k)^2 - 4 \times 2(6k+3) (= 2k^2 - 6k - 3 < 0)$$

(A1)

attempt to find the critical values of their quadratic equation or inequality

(M1)

$$k = 3.43649... \text{ OR } k = -0.436491... \text{ OR } k = \frac{3 \pm \sqrt{15}}{2}$$

$$k > 3.44 \left(= \frac{3 + \sqrt{15}}{2} \right) \text{ (since } k \text{ is positive)}$$

A1

Note: Accept $k \geq 3.44$.

[4 marks](c) $x = -2$ **A1****[1 mark]**

continued...

Question 12 continued.

(d) **METHOD 1**

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \dots \quad (\text{A1})$$

attempts division on $\frac{2x^2 + 6x - 3}{x + 2}$ **M1**

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \frac{2x - 3}{x + 2}$$

$$= 2x + 2 + \dots \quad (\text{A1})$$

asymptote is $y = 2x + 2$ **A1**

METHOD 2

let asymptote be $y = ax + b$

$$\frac{2x^2 + 6x - 3}{x + 2} = ax + b + \frac{c}{x + 2} \quad (\text{M1})$$

$$2x^2 + 6x - 3 = (ax + b)(x + 2) + c$$

equates coefficients of x^2 and x : **(M1)**

$$a = 2 \quad (\text{A1})$$

$$6 = 2a + b$$

$$b = 2 \quad (\text{A1})$$

$$(y = 2x + 2)$$

continued...

Question 12 continued.

METHOD 3

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \dots$$

A1

$$\frac{2x^2 + 6x - 3}{x + 2} - 2x = \frac{2x - 3}{x + 2}$$

attempt to find the limit as $x \rightarrow \infty$

M1

$$\lim_{x \rightarrow \infty} \frac{2x - 3}{x + 2} = 2$$

A1

$$y = 2x + 2$$

A1

[4 marks]

continued...

Question 12 continued.

(e) **METHOD 1**

EITHER

$$\text{for } k = 2, \frac{dy}{dx} = \frac{2x^2 + 8x + 15}{(x+2)^2} \quad \mathbf{A1}$$

$$\text{attempt to write } \frac{dy}{dx} \text{ in the form } p + \frac{q}{(x+2)^2} \text{ OR to write } \frac{dy}{dx} - 2 \text{ in the form } \frac{q}{(x+2)^2} \quad \mathbf{M1}$$

OR

$$y = 2x + 2 - \frac{7}{x+2} \quad \mathbf{A1}$$

Note: Follow through from their part (d).

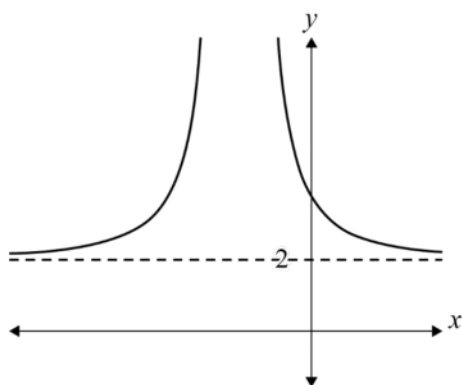
$$\text{attempt to find } \frac{dy}{dx} \text{ in the form } \frac{dy}{dx} = p + \frac{q}{(x+2)^2} \quad \mathbf{M1}$$

THEN

$$\frac{dy}{dx} = 2 + \frac{7}{(x+2)^2} \text{ OR } \frac{dy}{dx} - 2 = \frac{7}{(x+2)^2} \quad \mathbf{A1}$$

$$\text{since } \frac{7}{(x+2)^2} > 0 \text{ for } x \in \mathbb{R}, x \neq -2, \frac{dy}{dx} > 2. \quad \mathbf{R1}$$

Note: Award a maximum of **A1M1A0R0** for a graphical approach which includes a horizontal asymptote at 2, showing that $\frac{dy}{dx} > 2$.



continued...

Question 12 continued.

METHOD 2

for $k = 2$, $\frac{dy}{dx} = \frac{2x^2 + 8x + 15}{(x+2)^2}$ **A1**

clear assumption that $\frac{2x^2 + 8x + 15}{(x+2)^2} \leq 2$ **M1**

$(2x^2 + 8x + 15 \leq 2(x+2)^2 \Rightarrow) 15 \leq 8$ **A1**

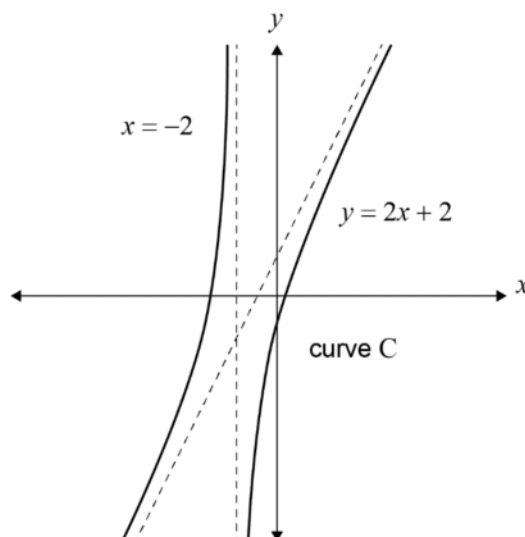
contradiction implies that $\frac{dy}{dx} > 2$ **R1**

Note: Award **A1M0A0R0** for solutions which work backwards from

$$\frac{2x^2 + 8x + 15}{(x+2)^2} > 2 \text{ to arrive at } 15 > 8.$$

[4 marks]

(f)



curve C in approx correct place, with negative y -intercept and part of the curve in 4th quadrant

A1

Note: The following **A1** marks should be awarded independently **provided** there is an attempt to draw both branches of $y = f(x)$.

vertical asymptote in approx correct place **A1**

oblique asymptote in approx correct place **A1**

approx correct asymptotic behaviour of C relative to both asymptotes **A1**

[4 marks]

Total [21 marks]

20 - (MathAA/21_HL_Summer_2025_Q2) - Trigonometry, Algebra-functions - Roots, Graphs

(a) period is $\frac{\pi}{2} (= 1.57079... = 1.57)$

A1**[1 mark]**

(b) attempt to substitute $x = \frac{\pi}{12}, f(x) = 5$ and $x = \frac{\pi}{3}, f(x) = 7$ to obtain two equations (**M1**)

Note: accept work where x values have been converted into degrees

$$a \tan\left(\frac{\pi}{6}\right) + b = 5 \text{ and } a \tan\left(\frac{2\pi}{3}\right) + b = 7 \left(\Rightarrow \frac{a}{\sqrt{3}} + b = 5 \text{ and } -a\sqrt{3} + b = 7 \right)$$

$$a = -\frac{\sqrt{3}}{2} (= -0.866025... = -0.866)$$

A1

$$b = \frac{11}{2} (= 5.5)$$

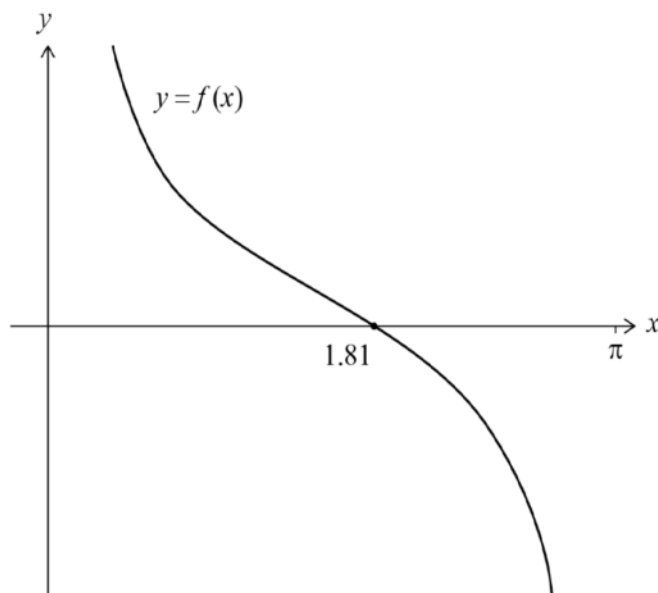
A1**Note:** These **A1** marks may be awarded independently.**[3 marks]****Total [4 marks]**

21 - (MathAA/21_HL_Summer_2025_Q6) - Trigonometry, Algebra-functions - Roots, Graphs

(a) (i) $(f(x) = \frac{4\cos x}{\sin x} + \sin x$

A1

(ii)



correct domain and curvature

A1

Note: The correct domain may be implied by a single branch of the function.

Condone the absence of π on the x -axis.

Award **A0** if the graph is not a function.

correct x -intercept at $x = 1.80911... = 1.81$

A1

Note: Accept coordinates $(1.81, 0)$. This mark is independent of the first **A1**.

[3 marks]

(b) $f^{-1}(2) = 1.31837...$
 $= 1.32$

A1

[1 mark]

(c) **METHOD 1****EITHER**

$$\text{recognition that } \sec x = \frac{1}{\cos x} \quad (M1)$$

$$\cos \alpha = \frac{2}{3} (= 0.666666\dots)$$

$$\alpha = \arccos \frac{2}{3} (= 0.841068\dots) \quad (A1)$$

OR

$$\alpha = \operatorname{arcsec} 1.5 (= 0.841068\dots) \quad (A2)$$

THEN

$$f(\alpha) = f(0.841068\dots) = 4.32306\dots$$

$$= 4.32 \left(= \frac{29\sqrt{5}}{15} \right) \quad A1$$

METHOD 2

$$\text{recognition that } \sec x = \frac{1}{\cos x} \quad (M1)$$

$$\cos \alpha = \frac{2}{3}, \sin \alpha = \frac{\sqrt{5}}{3} \left(\cot x = \frac{2}{\sqrt{5}} \right) \quad (A1)$$

$$f(\alpha) = \frac{4\left(\frac{2}{3}\right)}{\frac{\sqrt{5}}{3}} + \frac{\sqrt{5}}{3} \left(= 4\left(\frac{2}{\sqrt{5}}\right) + \frac{\sqrt{5}}{3} \right)$$

$$= \frac{29\sqrt{5}}{15} (= 4.32306\dots = 4.32) \quad A1$$

[3 marks]**Total [7 marks]**

22 - (MathAA/21_HL_Summer_2025_Q12) - Sequences - Series, Algebra-functions - Roots, Integration

(a) attempt to replace x with $-x$ **M1**

$$f_n(-x) = \sum_{r=0}^n (-2(-x)^2)^r \quad \text{OR} \quad -2(-x)^2 = -2x^2 \quad (\text{seen anywhere}) \quad \textbf{A1}$$

$$f_n(-x) = f_n(x) \quad \textbf{A1}$$

so f_n is even for all values of n **AG****[3 marks]**

$$(b) \quad (i) \quad f_3(x) = 1 - 2x^2 + (-2x^2)^2 + (-2x^2)^3 \quad \textbf{A1}$$

$$= 1 - 2x^2 + 4x^4 - 8x^6 \quad \textbf{AG}$$

$$(ii) \quad = 1 - 2x^2 + 4x^4 - 8x^6 + 16x^8 \quad \textbf{A1}$$

[2 marks](c) (i) recognition of geometric series with common ratio $-2x^2$ **(M1)**

$$\text{converges for } |-2x^2| < 1 \left(\Rightarrow x^2 < \frac{1}{2} \right) \quad \textbf{(A1)}$$

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \left(\Rightarrow -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2} \right)$$

$$\text{largest } K = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2} \right) \quad \textbf{A1}$$

(ii) use of formula for S_∞ of a geometric series with first term 1, common ratio $-2x^2$ **(M1)**

$$f(x) = \frac{1}{1 - (-2x^2)}$$

$$(f(x) =) \frac{1}{1 + 2x^2} \quad \textbf{A1}$$

$$a = 1, b = 2$$

[5 marks]

(d) (i) g is a one-to-one function (A1)

since g is a (strictly) decreasing function OR g has no points of zero gradient (turning points) R1

(ii) attempt to rearrange and swop x and y (at any stage) (M1)

$$x = \frac{1}{1+2y^2} \Rightarrow x + 2xy^2 = 1$$

$$y^2 = \frac{1-x}{2x} \quad \text{(A1)}$$

$$y = \pm \sqrt{\frac{1-x}{2x}}$$

$$g^{-1}(x) = \sqrt{\frac{1-x}{2x}} \quad \text{A1}$$

Note: Award **A0** if $g^{-1}(x)$ is missing.

Award **FT A1** marks only if f is of the correct form $f(x) = \frac{1}{a+bx^2}$

$$\text{(domain)} \quad \frac{1}{2} < x \leq 1 \quad \text{A1}$$

Note: This **A1** can be awarded independently.

[6 marks]

(e)

Note: Throughout part (e), do not award **A1** marks as **FT** from part (d)

METHOD 1

curves intersect at $x = 0.5897545107...$

(A1)

attempt to add the areas to the left and to the right of the point of intersection

(M1)

$$\int_0^{0.589...} g(x)dx + \int_{0.589...}^1 g^{-1}(x)dx = \int_0^{0.589...} \frac{1}{1+2x^2}dx + \int_{0.589...}^1 \sqrt{\frac{1-x}{2x}}dx$$

$$= 0.491548... + 0.143738...$$

(A1)

Note: Award **A1** for one correct value seen, dependent on **(M1)**

$$= 0.635286...$$

Area of $R = 0.635$

A1**METHOD 2**

curves intersect at $x = 0.5897545107...$

(A1)

attempt to find the area between $y = g(x)$ and $y = x$ to the left of the point of intersection

(M1)

$$\int_0^{0.589...} g(x)dx - \int_0^{0.589...} xdx = \int_0^{0.589...} \left(\frac{1}{1+2x^2} - x \right) dx$$

$$= 0.317643...$$

(A1)

$$2(0.3176432617...) = 0.635286...$$

Area of $R = 0.635$

A1

METHOD 3

curves intersect at $x = 0.5897545107...$

A1

attempt to find the area under $y = g^{-1}(x)$ and $y = x$ to the right of the point of intersection and the area of a square of side $x = 0.589...$

(M1)

$$0.589...^2 + 2 \int_{0.589...}^1 g^{-1}(x) dx = 0.589...^2 + 2 \int_{0.589...}^1 \sqrt{\frac{1-x}{2x}} dx$$

$$= 0.347810... + 2(0.143738...)$$

A1

Note: Award **A1** for 0.143..seen, dependent on **(M1)**

$$= 0.635286...$$

Area of $R = 0.635$

A1**[4 marks]****Total [20 marks]**

23 - (MathAA/22_HL_Summer_2025_Q12) - Differentiation, Algebra-functions - Roots, Integration

(a) **METHOD 1**

attempt to differentiate implicitly

(M1)

$$8x + 2y \frac{dy}{dx} - 24 + 4 \frac{dy}{dx} (= 0)$$

A1A1

Note: Award **A1** for $8x - 24$ and **A1** for $2y \frac{dy}{dx} + 4 \frac{dy}{dx}$.

Award **A1A0** for $8x + 2y \frac{dy}{dx} - 24 + 4 \frac{dy}{dx} = -20$ (or equivalent).

The implicit differentiation must be seen to award the above **A** marks.

$$\frac{dy}{dx} = \frac{24 - 8x}{2y + 4} \left(= \frac{12 - 4x}{y + 2} \right)$$

A1

$$\frac{dy}{dx} = \frac{4(3 - x)}{y + 2}$$

AG

METHOD 2

$$\frac{d}{dx}(4x^2 + y^2 - 24x + 4y + 20) = 8x - 24$$

A1

$$\frac{d}{dy}(4x^2 + y^2 - 24x + 4y + 20) = 2y + 4$$

A1

$$\frac{dy}{dx} = - \frac{\frac{dC}{dx}}{\frac{dC}{dy}}$$

(M1)

$$\frac{dy}{dx} = \frac{24 - 8x}{2y + 4} \left(= \frac{12 - 4x}{y + 2} \right)$$

A1

$$\frac{dy}{dx} = \frac{4(3 - x)}{y + 2}$$

AG

[4 marks]

(b) recognising boundaries of domain when $\frac{dy}{dx}$ undefined (M1)

$$y + 2 = 0$$

$$y = -2 \quad (A1)$$

substituting their y value into curve equation and attempt to solve for x (M1)

$$4x^2 + (-2)^2 - 24x + 4(-2) + 20 = 0 \quad (x^2 - 6x + 4 = 0)$$

$$x = \frac{6 \pm \sqrt{20}}{2} \text{ OR } (x-3)^2 - 5 = 0$$

$$3 - \sqrt{5} \leq x \leq 3 + \sqrt{5} \text{ (accept } a = 5 \text{)} \quad A1$$

Note: Award a maximum of **M1A1M1A0** for a final answer of $0.764 \leq x \leq 5.24$.

[4 marks]

(c) $x_A = 3$ A1

substituting their x value into curve equation (M1)

$$4(3)^2 + y^2 - 24(3) + 4y + 20 = 0$$

$$2.472135... \text{ OR } -2 + 2\sqrt{5}$$

$$y_A = 2.47 \text{ OR } y_A = -2 + 2\sqrt{5} \quad A1$$

$$A(3, 2.47) \text{ OR } A(3, -2 + 2\sqrt{5})$$

Note: Award **A1(M1)A1** for a final answer of $A(3, 2.5)$ with no working shown.

[3 marks]

(d) $4x^2 + (mx)^2 - 24x + 4(mx) + 20 = 0 \quad ((4 + m^2)x^2 + (4m - 24)x + 20 = 0)$ (A1)

$$(4m - 24)^2 - 4(4 + m^2)(20) \quad A1$$

recognising discriminant = 0 (seen anywhere) M1

$$\text{both } m = -4 \text{ and } m = 1 \quad A1$$

[4 marks]

(e) **METHOD 1**

attempt to form an equation in y by substituting $x = -\frac{1}{4}y$ **(M1)**

$$4\left(-\frac{1}{4}y\right)^2 + y^2 - 24\left(-\frac{1}{4}y\right) + 4y + 20 = 0 \quad \text{A1}$$

$$y_B = -4 \quad \text{A1}$$

Note: Award **(M1)A1A0** if an extra value is given for y_B .

METHOD 2

attempt to form an equation in x by substituting $y = -4x$ **(M1)**

$$4x^2 + (-4x)^2 - 24x + 4(-4x) + 20 = 0$$

$$x = 1 \quad \text{A1}$$

$$y_B = -4 \quad \text{A1}$$

Note: Award **(M1)A1A0** if an extra value is given for y_B .

[3 marks]

(f) attempt to solve for x by quadratic formula or completing the square **(M1)**

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4)(y^2 + 4y + 20)}}{2 \times 4} \quad \text{OR} \quad 4(x-3)^2 - 36 = -y^2 - 4y - 20$$

$$x = \frac{24 - \sqrt{256 - 64y - 16y^2}}{8} \left(x = 3 - \sqrt{\frac{16 - 4y - y^2}{4}} \right) \quad \text{A1}$$

Note: Condone $\pm$ at this stage.

$$V = \pi \int_{-4}^{-2+2\sqrt{5}} \left(\frac{24 - \sqrt{256 - 64y - 16y^2}}{8} \right)^2 dy \left(= \pi \int_{-4}^{2.47...} \left(3 - \sqrt{\frac{16 - 4y - y^2}{4}} \right)^2 dy \right) \quad \text{A1}$$

$$29.708027...$$

$$\text{volume} = 29.7 \quad \text{A1}$$

Note: Award **(M1)A1A1** for a final answer of 30 with no working shown.

[4 marks]

Total [22 marks]

24 - (MathAA/23_HL_Winter_2025_Q6) - Algebra-functions - Roots, Quadratics, Remainder & Factor Theorem

attempt to use remainder theorem $P(-1) = 12$ or to divide $P(x)$ by $(x+1)$ (M1)

$$2 - q + r = 12 \quad \text{A1}$$

$$(r - q = 10)$$

EITHER

$$\alpha + \beta = -\frac{q}{2} \text{ and } \alpha\beta = \frac{r}{2} \quad \text{(A1)}$$

attempt to use $(\alpha + \beta)^2 \equiv \alpha^2 + \beta^2 + 2\alpha\beta$ with $\alpha + \beta = -\frac{q}{2}$ and $\alpha\beta = \frac{r}{2}$ (M1)

$$\left(-\frac{q}{2}\right)^2 = \frac{37}{4} + 2\left(\frac{r}{2}\right) \quad \text{(A1)}$$

$$37 = q^2 - 4r$$

attempt to solve two simultaneous equations in q and r by rearranging/eliminating a variable (M1)

$$r = 10 + q \text{ and } r = \frac{q^2 - 37}{4} \text{ OR } q = r - 10 \text{ and } q = \sqrt{37 + 4r} \text{ OR } 37 = q^2 - 4(q + 10) \text{ OR}$$

$$37 = (r - 10)^2 - 4r$$

OR

recognition that α and β are both roots of $P(x)$ and adding the two equations (M1)

$$2\alpha^2 + q\alpha + r = 0 \text{ and } 2\beta^2 + q\beta + r = 0 \text{ then } 2(\alpha^2 + \beta^2) + q(\alpha + \beta) + 2r = 0$$

$$\alpha + \beta = -\frac{q}{2} \quad \text{(A1)}$$

use of $\alpha + \beta = -\frac{q}{2}$ and $\alpha^2 + \beta^2 = \frac{37}{4}$ to form equation in q (M1)

$$2 \times \frac{37}{4} - \frac{1}{2}q^2 + 20 + 2q = 0 \Rightarrow -\frac{1}{2}q^2 + 2q + \frac{77}{2} = 0 \quad \text{(A1)}$$

THEN

$$q = -7, r = 3$$

A1**Note:**

Award **A0** if extra values for q and r given without any indication that $q = -7, r = 3$ are chosen as their final answer.

Award a maximum of **(M1)A1(A1)(M1)(A1)(M1)A0** if final answer given as $q = 11, r = 21$

[7 marks]

25 - (MathAA/23_HL_Winter_2025_Q9) - Algebra-functions - Roots

$$\text{assume } \sqrt{2xy} > \frac{1}{2}(2x + y)$$

M1

Note: A statement of assumption of truth is required for example “assume”, “presume”, “suppose” or “let ... be true”.

Award **M0** for use of ‘let’ only or “consider”.

attempt to square both sides (which is valid since both x, y are positive)

M1

Note: This **M1** is independent of the previous method mark.

$$2xy > \frac{1}{4}(2x + y)^2$$

$$8xy > 4x^2 + 4xy + y^2$$

$$0 > 4x^2 - 4xy + y^2 \text{ OR } 0 > x^2 - xy + \frac{y^2}{4}$$

A1

$$0 > (2x - y)^2 \text{ OR } 0 > \left(x - \frac{y}{2}\right)^2$$

A1

a square number can never be negative hence contradiction

R1

Note: Do not award **A0R1**

$$\text{therefore } \sqrt{2xy} \leq \frac{1}{2}(2x + y)$$

AG**[5 marks]**

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