A LEVEL Cambridge Topical Past Papers

# **FURTHER** MATHEMATICS
# M**ECHAN**ICS P3

2020 — 2025




EXAMINENT
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1 - (9231/31_Summer_2020_Q6)

A particle $P$ of mass $m$ is moving with speed $u$ on a fixed smooth horizontal surface. The particle strikes a fixed vertical barrier. At the instant of impact the direction of motion of $P$ makes an angle $\alpha$ with the barrier. The coefficient of restitution between $P$ and the barrier is $e$. As a result of the impact, the direction of motion of $P$ is turned through 90°.

**(a)** Show that $\tan^2\alpha = \dfrac{1}{e}$. [3]

The particle $P$ loses two-thirds of its kinetic energy in the impact.

**(b)** Find the value of $\alpha$ and the value of $e$. [5]

2 - (9231/32_Summer_2020_Q6)

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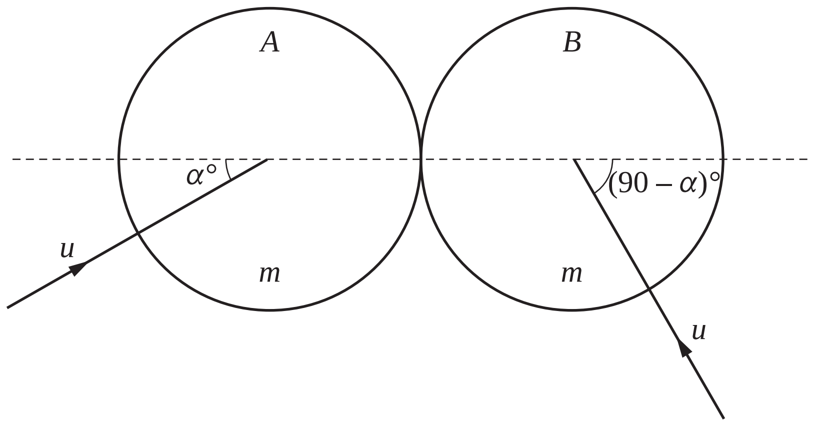
3 - (9231/33_Summer_2020_Q5)




Two uniform smooth spheres $A$ and $B$ of equal radii each have mass $m$. The two spheres are each moving with speed $u$ on a horizontal surface when they collide. Immediately before the collision $A$'s direction of motion makes an angle of $\alpha°$ with the line of centres, and $B$'s direction of motion is perpendicular to that of $A$ (see diagram). The coefficient of restitution between the spheres is $e$.

Immediately after the collision, $B$ moves in a direction at right angles to the line of centres.

**(a)** Show that $\tan\alpha = \dfrac{1+e}{1-e}$. [4]

**(b)** Given that $\tan\alpha = 2$, find the speed of $A$ after the collision. [4]

4 - (9231/31_Winter_2020_Q6)

Two smooth spheres $A$ and $B$ have equal radii and masses $m$ and $2m$ respectively. Sphere $B$ is at rest on a smooth horizontal floor. Sphere $A$ is moving on the floor with velocity $u$ and collides directly with $B$. The coefficient of restitution between the spheres is $e$.

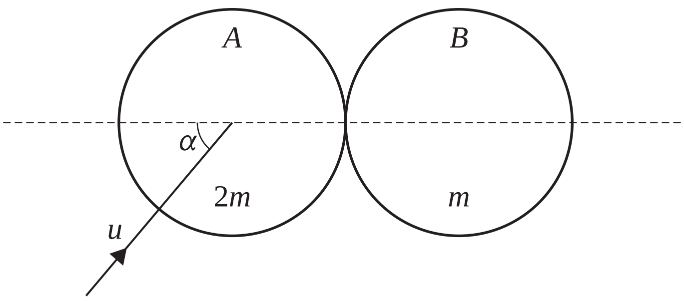
**(a)** Find, in terms of $u$ and $e$, the velocities of $A$ and $B$ after the collision. [3]

Subsequently, $B$ collides with a fixed vertical wall which makes an angle $\theta$ with the direction of motion of $B$, where $\tan\theta = \frac{3}{4}$.

The coefficient of restitution between $B$ and the wall is $\frac{2}{3}$. Immediately after $B$ collides with the wall, the kinetic energy of $A$ is $\frac{5}{32}$ of the kinetic energy of $B$.

**(b)** Find the possible values of $e$. [7]

5 - (9231/32_Winter_2020_Q2)




Two uniform smooth spheres $A$ and $B$ of equal radii have masses $2m$ and $m$ respectively. Sphere $B$ is at rest on a smooth horizontal surface. Sphere $A$ is moving on the surface with speed $u$ and collides with $B$. Immediately before the collision, the direction of motion of $A$ makes an angle $\alpha$ with the line of centres of the spheres, where $\tan\alpha = \frac{4}{3}$ (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

Find the speed of $A$ after the collision. [5]

6 - (9231/32_Winter_2020_Q4)

A particle $P$ of mass $m$ is moving in a horizontal circle with angular speed $\omega$ on the smooth inner surface of a hemispherical shell of radius $r$. The angle between the vertical and the normal reaction of the surface on $P$ is $\theta$.

**(a)** Show that $\cos\theta = \dfrac{g}{\omega^2 r}$. [3]

The plane of the circular motion is at a height $x$ above the lowest point of the shell. When the angular speed is doubled, the plane of the motion is at a height $4x$ above the lowest point of the shell.

**(b)** Find $x$ in terms of $r$. [4]

7 - (9231/33_Winter_2020_Q6)

Two smooth spheres $A$ and $B$ have equal radii and masses $m$ and $2m$ respectively. Sphere $B$ is at rest on a smooth horizontal floor. Sphere $A$ is moving on the floor with velocity $u$ and collides directly with $B$. The coefficient of restitution between the spheres is $e$.

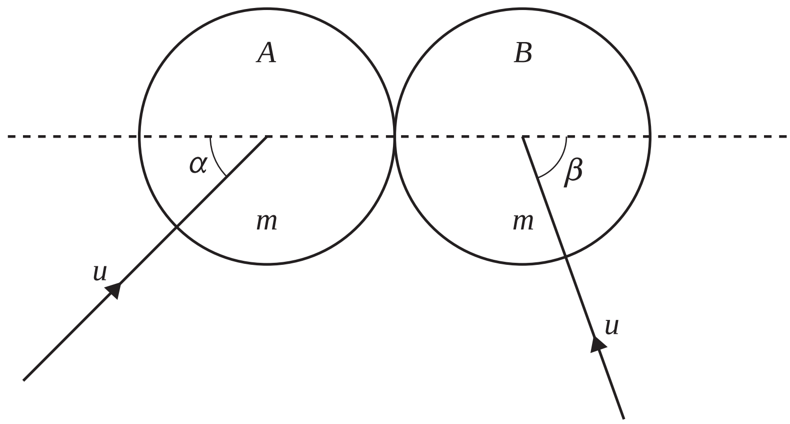
**(a)** Find, in terms of $u$ and $e$, the velocities of $A$ and $B$ after the collision. [3]

Subsequently, $B$ collides with a fixed vertical wall which makes an angle $\theta$ with the direction of motion of $B$, where $\tan\theta = \frac{3}{4}$.

The coefficient of restitution between $B$ and the wall is $\frac{2}{3}$. Immediately after $B$ collides with the wall, the kinetic energy of $A$ is $\frac{5}{32}$ of the kinetic energy of $B$.

**(b)** Find the possible values of $e$. [7]

8 - (9231/31_Summer_2021_Q6)



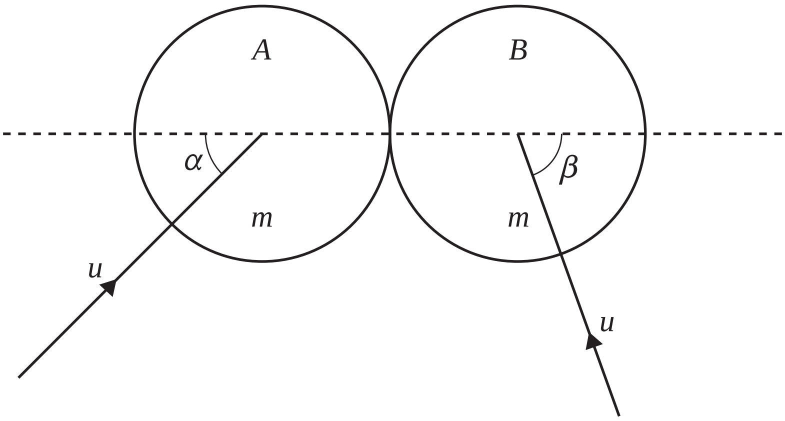

Two uniform smooth spheres $A$ and $B$ of equal radii each have mass $m$. The two spheres are each moving with speed $u$ on a horizontal surface when they collide. Immediately before the collision, $A$'s direction of motion makes an angle $\alpha$ with the line of centres, and $B$'s direction of motion makes an angle $\beta$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$ and $2\cos\beta = \cos\alpha$.

**(a)** Show that the direction of motion of $A$ after the collision is perpendicular to the line of centres. [4]

The total kinetic energy of the spheres after the collision is $\frac{3}{4}mu^2$.

**(b)** Find the value of $\alpha$. [4]

9 - (9231/32_Summer_2021_Q6)




Two uniform smooth spheres $A$ and $B$ of equal radii each have mass $m$. The two spheres are each moving with speed $u$ on a horizontal surface when they collide. Immediately before the collision, $A$'s direction of motion makes an angle $\alpha$ with the line of centres, and $B$'s direction of motion makes an angle $\beta$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$ and $2\cos\beta = \cos\alpha$.

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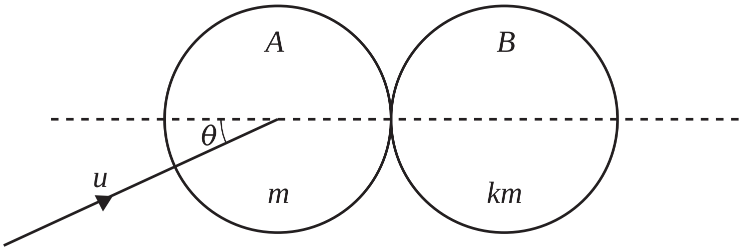
10 - (9231/33_Summer_2021_Q6)




Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $km$ respectively. Sphere $A$ is moving with speed $u$ on a smooth horizontal surface when it collides with sphere $B$ which is at rest. Immediately before the collision, $A$'s direction of motion makes an angle $\theta$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

**(a)** Show that the speed of $B$ after the collision is $\frac{4u\cos\theta}{3(1+k)}$. [3]

70% of the total kinetic energy of the spheres is lost as a result of the collision.

**(b)** Given that $\tan\theta = \frac{1}{3}$, find the value of $k$. [6]

11 - (9231/31_Winter_2021_Q2)

A particle $P$ of mass $m$ kg moves along a horizontal straight line with acceleration $a\,\mathrm{m\,s^{-2}}$ given by

$$a = \frac{v(1-2t^2)}{t},$$

where $v\,\mathrm{m\,s^{-1}}$ is the velocity of $P$ at time $t$ s.

**(a)** Find an expression for $v$ in terms of $t$ and an arbitrary constant. [3]

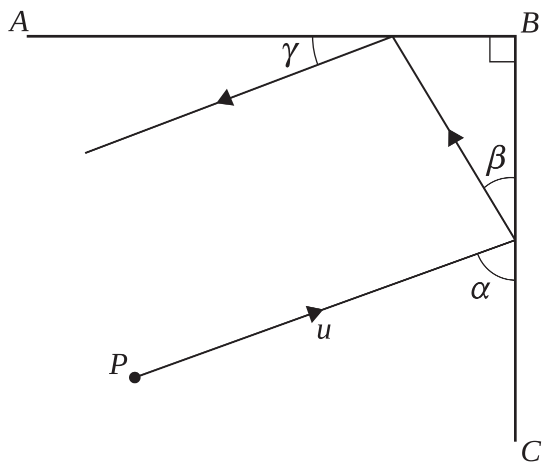
**(b)** Given that $a = 5$ when $t = 1$, find an expression, in terms of $m$ and $t$, for the horizontal force acting on $P$ at time $t$. [3]

12 - (9231/31_Winter_2021_Q5)

A particle $P$ is projected from a point $O$ on a horizontal plane and moves freely under gravity. Its initial speed is $u\,\mathrm{m\,s^{-1}}$ and its angle of projection is $\sin^{-1}(\frac{4}{5})$ above the horizontal. At time 8 s after projection, $P$ is at the point $A$. At time 32 s after projection, $P$ is at the point $B$. The direction of motion of $P$ at $B$ is perpendicular to its direction of motion at $A$.

Find the value of $u$. [7]

13 - (9231/31_Winter_2021_Q7)




The smooth vertical walls $AB$ and $CB$ are at right angles to each other. A particle $P$ is moving with speed $u$ on a smooth horizontal floor and strikes the wall $CB$ at an angle $\alpha$. It rebounds at an angle $\beta$ to the wall $CB$. The particle then strikes the wall $AB$ and rebounds at an angle $\gamma$ to that wall (see diagram). The coefficient of restitution between each wall and $P$ is $e$.

**(a)** Show that $\tan\beta = e\tan\alpha$. [3]

**(b)** Express $\gamma$ in terms of $\alpha$ and explain what this result means about the final direction of motion of $P$. [4]

As a result of the two impacts the particle loses $\frac{8}{9}$ of its initial kinetic energy.

**(c)** Given that $\alpha + \beta = 90°$, find the value of $e$ and the value of $\tan\alpha$. [4]

14 - (9231/32_Winter_2021_Q1)

A particle is projected with speed $u$ at an angle $\alpha$ above the horizontal from a point $O$ on a horizontal plane. The particle moves freely under gravity.

**(a)** Write down the horizontal and vertical components of the velocity of the particle at time $T$ after projection. [2]

At time $T$ after projection, the direction of motion of the particle is perpendicular to the direction of projection.

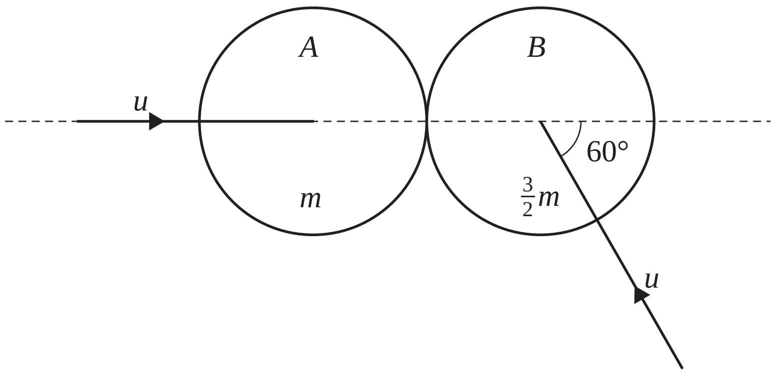
**(b)** Express $T$ in terms of $u$, $g$ and $\alpha$. [2]

At time $T$ after projection, the direction of motion of the particle is perpendicular to the direction of projection.

Express $T$ in terms of $u$, $g$ and $\alpha$. [2]

**(c)** Deduce that $T > \frac{u}{g}$. [1]

15 - (9231/32_Winter_2021_Q5)




Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $\frac{3}{2}m$ respectively. The two spheres are each moving with speed $u$ on a horizontal surface when they collide. Immediately before the collision $A$'s direction of motion is along the line of centres, and $B$'s direction of motion makes an angle of 60° with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$.

**(a)** Find the angle through which the direction of motion of $B$ is deflected by the collision. [6]

**(b)** Find the loss in the total kinetic energy of the system as a result of the collision. [3]

16 - (9231/32_Winter_2021_Q6)

A particle $P$ of mass 2 kg moves along a horizontal straight line. The point $O$ is a fixed point on this line. At time $t$ s the velocity of $P$ is $v\,\text{m s}^{-1}$ and the displacement of $P$ from $O$ is $x$ m.

A force of magnitude $\left(8x - \frac{128}{x^3}\right)$ N acts on $P$ in the direction $OP$. When $t = 0$, $x = 8$ and $v = -15$.

**(a)** Show that $v = -\frac{2}{x}(x^2 - 4)$. [5]

**(b)** Find an expression for $x$ in terms of $t$. [4]

17 - (9231/33_Winter_2021_Q2)

A particle $P$ of mass $m$ kg moves along a horizontal straight line with acceleration $a\,\text{m s}^{-2}$ given by

$$a = \frac{v(1-2t^2)}{t},$$

where $v\,\text{m s}^{-1}$ is the velocity of $P$ at time $t$ s.

**(a)** Find an expression for $v$ in terms of $t$ and an arbitrary constant. [3]

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18 - (9231/33_Winter_2021_Q5)

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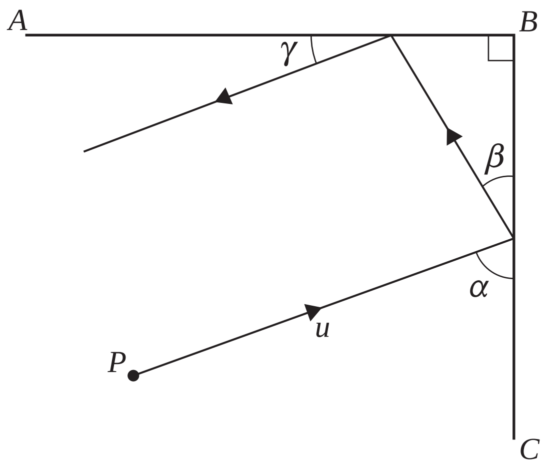
Find the value of $u$. [7]

19 - (9231/33_Winter_2021_Q7)



The smooth vertical walls $AB$ and $CB$ are at right angles to each other. A particle $P$ is moving with speed $u$ on a smooth horizontal floor and strikes the wall $CB$ at an angle $\alpha$. It rebounds at an angle $\beta$ to the wall $CB$. The particle then strikes the wall $AB$ and rebounds at an angle $\gamma$ to that wall (see diagram). The coefficient of restitution between each wall and $P$ is $e$.

**(a)** Show that $\tan\beta = e\tan\alpha$. [3]

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As a result of the two impacts the particle loses $\frac{8}{9}$ of its initial kinetic energy.

**(c)** Given that $\alpha + \beta = 90°$, find the value of $e$ and the value of $\tan\alpha$. [4]

20 - (9231/31_Summer_2022_Q6)

Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $km$ respectively. The two spheres are on a horizontal surface. Sphere $A$ is travelling with speed $u$ towards sphere $B$ which is at rest. The spheres collide. Immediately before the collision, the direction of motion of $A$ makes an angle $\alpha$ with the line of centres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

**(a)** Show that the speed of $B$ after the collision is $\dfrac{3u\cos\alpha}{2(1+k)}$ and find also an expression for the speed of $A$ along the line of centres after the collision, in terms of $k$, $u$ and $\alpha$. [4]

After the collision, the kinetic energy of $A$ is equal to the kinetic energy of $B$.

**(b)** Given that $\tan\alpha = \frac{2}{3}$, find the possible values of $k$. [5]

21 - (9231/31_Summer_2022_Q7)

Particles $P$ and $Q$ are projected in the same vertical plane from a point $O$ at the top of a cliff. The height of the cliff exceeds 50 m. Both particles move freely under gravity. Particle $P$ is projected with speed $\frac{35}{2}\,\mathrm{m\,s^{-1}}$ at an angle $\alpha$ above the horizontal, where $\tan\alpha = \frac{4}{3}$. Particle $Q$ is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle $\beta$ above the horizontal, where $\tan\beta = \frac{1}{2}$. Particle $Q$ is projected one second after the projection of particle $P$. The particles collide $T$ s after the projection of particle $Q$.

**(a)** Write down expressions, in terms of $T$, for the horizontal displacements of $P$ and $Q$ from $O$ when they collide and hence show that $4uT = 21\sqrt{5}(T+1)$. [4]

**(b)** Find the value of $T$. [4]

**(c)** Find the horizontal and vertical displacements of the particles from $O$ when they collide. [3]

22 - (9231/32_Summer_2022_Q6)

Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $km$ respectively. The two spheres are on a horizontal surface. Sphere $A$ is travelling with speed $u$ towards sphere $B$ which is at rest. The spheres collide. Immediately before the collision, the direction of motion of $A$ makes an angle $\alpha$ with the line of centres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

**(a)** Show that the speed of $B$ after the collision is $\dfrac{3u\cos\alpha}{2(1+k)}$ and find also an expression for the speed of $A$ along the line of centres after the collision, in terms of $k$, $u$ and $\alpha$. [4]

After the collision, the kinetic energy of $A$ is equal to the kinetic energy of $B$.

**(b)** Given that $\tan\alpha = \frac{2}{3}$, find the possible values of $k$. [5]

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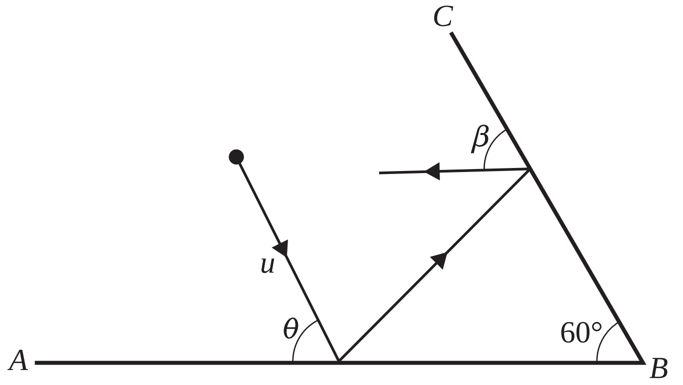
**(c)** Find the horizontal and vertical displacements of the particles from $O$ when they collide. [3]

24 - (9231/33_Summer_2022_Q6)




$AB$ and $BC$ are two fixed smooth vertical barriers on a smooth horizontal surface, with angle $ABC = 60°$. A particle of mass $m$ is moving with speed $u$ on the surface. The particle strikes $AB$ at an angle $\theta$ with $AB$. It then strikes $BC$ and rebounds at an angle $\beta$ with $BC$ (see diagram). The coefficient of restitution between the particle and each barrier is $e$ and $\tan\theta = 2$.

The kinetic energy of the particle after the first collision is 40% of its kinetic energy before the first collision.

**(a)** Find the value of $e$. [4]

**(b)** Find the size of angle $\beta$. [4]

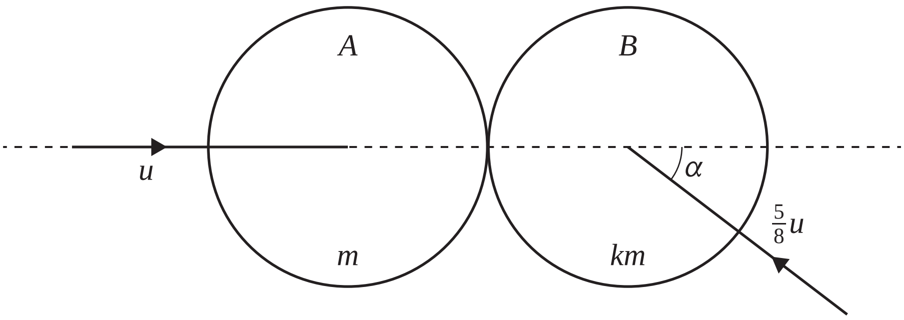
25 - (9231/31_Winter_2022_Q4)

A particle of mass 0.5 kg moves along a horizontal straight line. Its velocity is $v\,\mathrm{m\,s^{-1}}$ at time $t$ s. The forces acting on the particle are a driving force of magnitude 50 N and a resistance of magnitude $2v^2$ N. The initial velocity of the particle is $3\,\mathrm{m\,s^{-1}}$.

**(a)** Find an expression for $v$ in terms of $t$. [7]

**(b)** Deduce the limiting value of $v$. [1]

26 - (9231/31_Winter_2022_Q6)




Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $km$ respectively. The two spheres are moving on a horizontal surface with speeds $u$ and $\frac{5}{8}u$ respectively. Immediately before the spheres collide, $A$ is travelling along the line of centres, and $B$'s direction of motion makes an angle $\alpha$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$ and $\tan\alpha = \frac{3}{4}$.

After the collision, the direction of motion of $B$ is perpendicular to the line of centres.

**(a)** Find the value of $k$. [4]

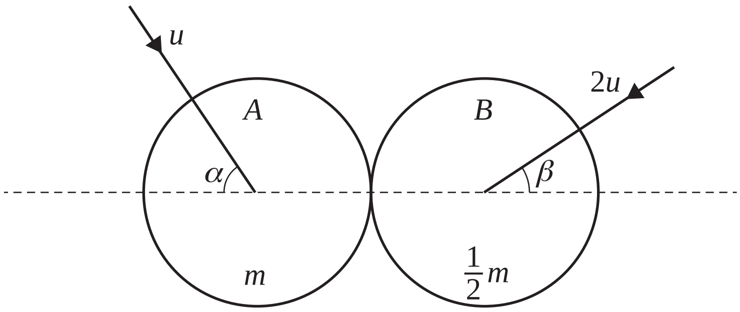
**(b)** Find the loss in the total kinetic energy as a result of the collision. [4]

27 - (9231/32_Winter_2022_Q3)

One end of a light elastic string, of natural length $a$ and modulus of elasticity $\frac{16}{3}Mg$, is attached to a fixed point $O$. A particle $P$ of mass $4M$ is attached to the other end of the string and hangs vertically in equilibrium. Another particle of mass $2M$ is attached to $P$ and the combined particle is then released from rest. The speed of the combined particle when it has descended a distance $\frac{1}{4}a$ is $v$.

Find an expression for $v$ in terms of $g$ and $a$. [6]

28 - (9231/32_Winter_2022_Q7)



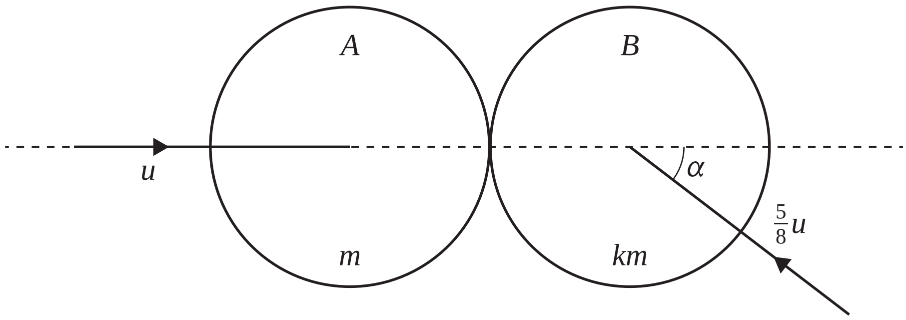

Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $\frac{1}{2}m$ respectively. The two spheres are moving on a horizontal surface when they collide. Immediately before the collision, sphere $A$ is travelling with speed $u$ and its direction of motion makes an angle $\alpha$ with the line of centres. Sphere $B$ is travelling with speed $2u$ and its direction of motion makes an angle $\beta$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{5}{8}$ and $\alpha + \beta = 90°$.

**(a)** Find the component of the velocity of $B$ parallel to the line of centres after the collision, giving your answer in terms of $u$ and $\alpha$. [4]

The direction of motion of $B$ after the collision is parallel to the direction of motion of $A$ before the collision.

**(b)** Find the value of $\tan\alpha$. [5]

29 - (9231/33_Winter_2022_Q6)



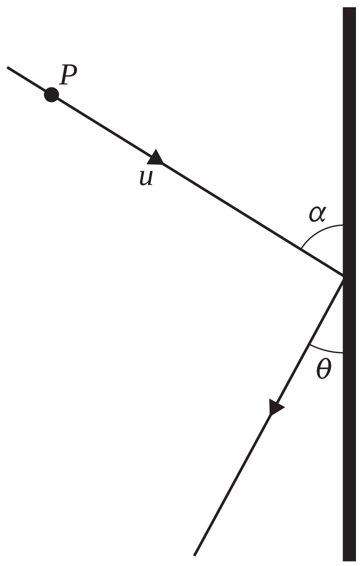
Two uniform smooth spheres $A$ and $B$ of equal radii have masses $m$ and $km$ respectively. The two spheres are moving on a horizontal surface with speeds $u$ and $\frac{5}{8}u$ respectively. Immediately before the spheres collide, $A$ is travelling along the line of centres, and $B$'s direction of motion makes an angle $\alpha$ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$ and $\tan\alpha = \frac{3}{4}$.

After the collision, the direction of motion of $B$ is perpendicular to the line of centres.

**(a)** Find the value of $k$. [4]

**(b)** Find the loss in the total kinetic energy as a result of the collision. [4]

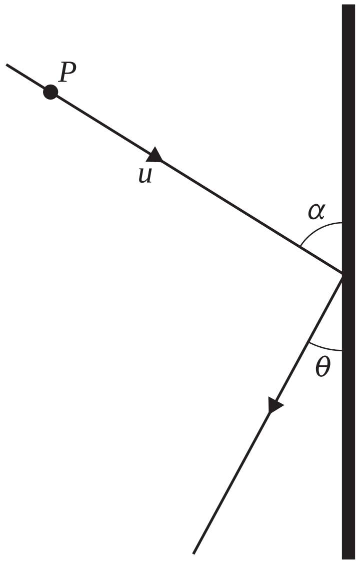
30 - (9231/31_Summer_2023_Q2)




A particle $P$ of mass $m$ is moving with speed $u$ on a fixed smooth horizontal surface. It collides at an angle α with a fixed smooth vertical barrier. After the collision, $P$ moves at an angle $\theta$ with the barrier, where $\tan\theta = \frac{1}{2}$ (see diagram). The coefficient of restitution between $P$ and the barrier is $e$. The particle $P$ loses 20% of its kinetic energy as a result of the collision.

Find the value of $e$. [5]

31 - (9231/32_Summer_2023_Q2)

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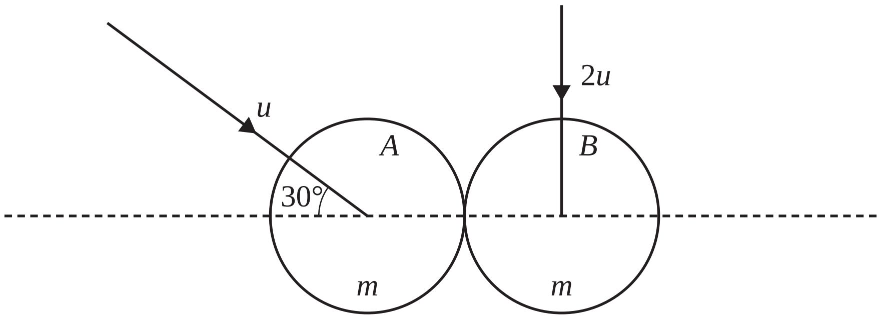
Find the value of $e$. [5]

32 - (9231/33_Summer_2023_Q4)




Two identical smooth uniform spheres $A$ and $B$ each have mass $m$. The two spheres are moving on a smooth horizontal surface when they collide with speeds $u$ and $2u$ respectively. Immediately before the collision, $A$'s direction of motion makes an angle of 30° with the line of centres, and $B$'s direction of motion is perpendicular to the line of centres (see diagram). After the collision, $A$ and $B$ are moving in the same direction. The coefficient of restitution between the spheres is $e$.

**(a)** Find the value of $e$. [5]

**(b)** Find the loss in the total kinetic energy of the spheres as a result of the collision. [3]

1 - (9231/31_Summer_2020_Q6)

| | | |
|---|---|---|
| (a) | Let components of velocity (parallel to plane and perpendicular) after impact be $(x, y)$<br>$y = v\cos\alpha = eu\sin\alpha$ | B1 |
| | $x = v\sin\alpha = u\cos\alpha$ | B1 |
| | Divide: $\tan\alpha = \frac{1}{e\tan\alpha}$ : $\tan^2\alpha = \frac{1}{e}$. | B1 |
| | | 3 |
| (b) | $v^2 = \frac{1}{3}u^2$ | B1 |
| | $\left(\frac{u\cos\alpha}{\sin\alpha}\right)^2 = \frac{1}{3}u^2$ | M1 |
| | $(\tan\alpha)^2 = 3$ | M1 |
| | $\alpha = 60°$ | A1 |
| | $e = \frac{1}{3}$ | A1 |

| | | |
|---|---|---|
| | Alternative method for **(b)** | |
| | KE after impact = $\frac{1}{2}m\left(x^2+y^2\right)=\frac{1}{2}m\left((u\cos\alpha)^2+e^2(u\sin\alpha)^2\right)$ | **M1** |
| | From **(a)** $\sin\alpha=1/\sqrt{(1+e)}$ and $\cos\alpha=\sqrt{e}/\sqrt{(1+e)}$ | **B1** |
| | KE = $\frac{1}{2}mu^2\left(\frac{e}{1+e}+\frac{e^2}{1+e}\right)$ = $\frac{1}{2}mu^2e$ | **A1** |
| | This is equal to $\frac{1}{3}\times\frac{1}{2}mu^2$ so $e=\frac{1}{3}$ | **M1** |
| | $\tan\alpha=\sqrt{3},\ \alpha=60°$ | **A1** |
| | | **5** |

2 - (9231/32_Summer_2020_Q6)

| | | |
|---|---|---|
| (a) | Let components of velocity (parallel to plane and perpendicular) after impact be $(x, y)$<br>$y=v\cos\alpha=eu\sin\alpha$ | **B1** |
| | $x=v\sin\alpha=u\cos\alpha$ | **B1** |
| | Divide: $\tan\alpha=\frac{1}{e\tan\alpha}$ : $\tan^2\alpha$ = $\frac{1}{e}$. | **B1** |
| | | **3** |

| | | |
|---|---|---|
| (b) | $v^2 = \frac{1}{3}u^2$ | **B1** |
| | $\left(\frac{u\cos\alpha}{\sin\alpha}\right)^2 = \frac{1}{3}u^2$ | **M1** |
| | $(\tan\alpha)^2 = 3$ | **M1** |
| | $\alpha = 60°$ | **A1** |
| | $e = \frac{1}{3}$ | **A1** |
| | Alternative method for **(b)** | |
| | KE after impact = $\frac{1}{2}m\left(x^2+y^2\right) = \frac{1}{2}m\left((u\cos\alpha)^2 + e^2(u\sin\alpha)^2\right)$ | **M1** |
| | From **(a)** $\sin\alpha = 1/\sqrt{(1+e)}$ and $\cos\alpha = \sqrt{e}/\sqrt{(1+e)}$ | **B1** |
| | KE = $\frac{1}{2}mu^2\left(\frac{e}{1+e}+\frac{e^2}{1+e}\right)$ = $\frac{1}{2}mu^2e$ | **A1** |
| | This is equal to $\frac{1}{3}\times\frac{1}{2}mu^2$ so $e = \frac{1}{3}$ | **M1** |
| | $\tan\alpha = \sqrt{3},\ \alpha = 60°$ | **A1** |
| | | **5** |

3 - (9231/33_Summer_2020_Q5)

| | | |
|---|---|---|
| (a) | Let $w$ be speed of $A$ along line of centres after collision<br>$\leftarrow \quad mw = -mu\cos\alpha + mu\sin\alpha$ | **M1** |
| | $w - 0 = e(u\cos\alpha + u\sin\alpha)$ | **M1** |
| | Rearrange:<br>$\sin\alpha(u - eu) = \cos\alpha(u + eu)$ | **M1** |
| | $\tan\alpha = \dfrac{1+e}{1-e}$. **AG** | **A1** |
| | | **4** |
| (b) | $\tan\alpha = 2 \Rightarrow e = \dfrac{1}{3}$ | **B1** |
| | $w = \dfrac{1}{3}u\left(\dfrac{1}{\sqrt{5}} + \dfrac{2}{\sqrt{5}}\right) = \dfrac{u}{\sqrt{5}}$ | **M1** |
| | Speed = $\sqrt{w^2 + (u\sin\alpha)^2}$ | **M1** |
| | $= \sqrt{\dfrac{u^2}{5} + \dfrac{4u^2}{5}} = u$ | **A1** |
| | | **4** |

4 - (9231/31_Winter_2020_Q6)

| | | | |
|---|---|---|---|
| (a) | $mu = mw + 2mv$ | **B1** | Momentum equation (with m) |
| | $v - w = eu$ | **B1** | Restitution with consistent signs |
| | $v = \frac{u}{3}(e+1)$<br>$w = \frac{u}{3}(1-2e)$ | **B1** | Both correct. |
| | | **3** | |
| (b) | Perpendicular to plane: $y = ev\sin\theta$<br>Parallel to plane: $x = v\cos\theta$ | **B1** | Both |
| | Speed of $B = \sqrt{x^2 + y^2} = \sqrt{v^2(\left(\frac{4}{5}\right)^2 + \left(\frac{2}{3}.\frac{3}{5}\right)^2)}$ $(= \frac{2}{\sqrt{5}}v)$ | **M1** | Speed of B |
| | KE of $B = \frac{1}{2}.2m\frac{4}{5}.\frac{u^2}{9}(e+1)^2$ | **M1** | KE of B in terms of $u$. $\frac{1}{2}$ and $2m$ needed |
| | KE of $A = \frac{1}{2}.m.\frac{u^2}{9}(1-2e)^2$<br>So $\frac{1}{2}.m.\frac{u^2}{9}(1-2e)^2 = \frac{5}{32}.\frac{1}{2}.2m\frac{4}{5}.\frac{u^2}{9}(e+1)^2$ | **M1 A1** | Relate the two KEs |
| | $4(1-2e)^2 = (e+1)^2$ or $15e^2 - 18e + 3 = 0$ | **M1** | Rearrange and simplify to quadratic |
| | $1 + e = \pm 2(1-2e)$<br>$e = \frac{1}{5}, \ 1$ | **A1** | Both values |
| | | **7** | |

5 - (9231/32_Winter_2020_Q2)

| | | | |
|---|---|---|---|
| | Speeds $v$ and $w$ after collision<br>$2mv + mw = 2mu\cos\alpha$ | **M1** | Momentum equation with $m$. Correct masses, allow sin instead of cos |
| | $w - v = eu\cos\alpha$ | **M1** | Restitution, with consistent signs |
| | $v = \frac{1}{3}u\cos\alpha(2-e) = \frac{1}{3}u.\frac{3}{5}\left(2-\frac{1}{3}\right) = \frac{1}{3}u$ | **A1** | |
| | Square of speed of $A$ = $\left(\frac{1}{3}u\right)^2 + (u\sin\alpha)^2$ | **M1** | Uses correct speed perpendicular to motion |
| | $= \left(\frac{1}{3}u\right)^2 + \left(\frac{4}{5}u\right)^2$<br>Speed = $\frac{13}{15}u$ (= $0.867u$) | **A1** | |
| | | **5** | |

6 - (9231/32_Winter_2020_Q4)

| | | | |
|---|---|---|---|
| (a) | $\uparrow\ N\cos\theta = mg$ | **B1** | |
| | $\leftarrow\ N\sin\theta = mr\sin\theta\omega^2$ | **B1** | |
| | $\cos\theta = \frac{mg}{N}$ so $\cos\theta = \frac{g}{\omega^2 r}$ | **B1** | AG |
| | | **3** | |

| | | | |
|---|---|---|---|
| (b) | $\cos\theta = \frac{r-x}{r} = \frac{g}{\omega^2 r}$ | **B1** | Using trig of situation: must involve $x$ |
| | In new situation: $r - 4x = r \times \frac{g}{4\omega^2 r}$ | **M1** | Using new situation with $4x$ and $2\omega$ seen |
| | $r - x = 4(r - 4x)$ | **M1** | Combining |
| | $x = \frac{1}{5}r$ | **A1** | |
| | | **4** | |

7 - (9231/33_Winter_2020_Q6)

| | | | |
|---|---|---|---|
| (a) | $mu = mw + 2mv$ | **B1** | Momentum equation (with m) |
| | $v - w = eu$ | **B1** | Restitution with consistent signs |
| | $v = \frac{u}{3}(e+1)$<br>$w = \frac{u}{3}(1-2e)$ | **B1** | Both correct. |
| | | **3** | |

| | | | |
|---|---|---|---|
| (b) | Perpendicular to plane: $y = ev\sin\theta$<br>Parallel to plane: $x = v\cos\theta$ | **B1** | Both |
| | Speed of $B = \sqrt{x^2+y^2} = \sqrt{v^2\left(\left(\frac{4}{5}\right)^2+\left(\frac{2}{3}.\frac{3}{5}\right)^2\right)}$ $\left(= \frac{2}{\sqrt{5}}v\right)$ | **M1** | Speed of B |
| | KE of $B = \frac{1}{2}.2m\frac{4}{5}.\frac{u^2}{9}(e+1)^2$ | **M1** | KE of B in terms of $u$ . $\frac{1}{2}$ and $2m$ needed |
| | KE of $A = \frac{1}{2}.m.\frac{u^2}{9}(1-2e)^2$<br>So $\frac{1}{2}.m.\frac{u^2}{9}(1-2e)^2 = \frac{5}{32}.\frac{1}{2}.2m\frac{4}{5}.\frac{u^2}{9}(e+1)^2$ | **M1 A1** | Relate the two KEs |
| | $4(1-2e)^2 = (e+1)^2$ or $15e^2-18e+3=0$ | **M1** | Rearrange and simplify to quadratic |
| | $1+e=\pm 2(1-2e)$<br>$e=\frac{1}{5},\ 1$ | **A1** | Both values |
| | | **7** | |

8 - (9231/31_Summer_2021_Q6)

| | | | |
|---|---|---|---|
| (a) | Along line of centres, speeds $v_1$ and $v_2$ <br> $mv_1 + mv_2 = mu\cos\alpha - mu\cos\beta$ | **M1** | Momentum (condone missing masses). |
| | $v_2 - v_1 = eu(\cos\beta + \cos\alpha)$ | **M1** | Restitution. |
| | Both correct, masses seen. | **A1** | |
| | $v_1 = 0$ so *A* has no speed along line of centres: moves perpendicular to line of centres | **A1** | AG. |
| | | **4** | |
| (b) | $(v_2 = \frac{1}{2}u\ \cos\alpha = u\cos\beta\ )$ <br> KE of *B* after collision is $\frac{1}{2}m\left(v_2{}^2 + (u\sin\beta)^2\right)$ <br> KE of *A* after collision = $\frac{1}{2}m(u\sin\alpha)^2$ | **M1** | Both components. |
| | Add both KEs and equate to $\frac{3}{4}mu^2$ | **M1** | |
| | Simplify to equation in $\sin\alpha$ | **M1** | |
| | $\sin\alpha = \frac{1}{\sqrt{2}}$, $\alpha = 45°$ | **A1** | |
| | | **4** | |

9 - (9231/32_Summer_2021_Q6)

| | | | |
|---|---|---|---|
| (a) | Along line of centres, speeds $v_1$ and $v_2$ <br> $mv_1 + mv_2 = mu\cos\alpha - mu\cos\beta$ | **M1** | Momentum (condone missing masses). |
| | $v_2 - v_1 = eu(\cos\beta + \cos\alpha)$ | **M1** | Restitution. |
| | Both correct, masses seen. | **A1** | |
| | $v_1 = 0$ so *A* has no speed along line of centres: moves perpendicular to line of centres | **A1** | AG. |
| | | **4** | |
| (b) | $(v_2 = \frac{1}{2}u\cos\alpha = u\cos\beta)$ <br> KE of *B* after collision is $\frac{1}{2}m\left(v_2{}^2 + (u\sin\beta)^2\right)$ <br> KE of *A* after collision = $\frac{1}{2}m(u\sin\alpha)^2$ | **M1** | Both components. |
| | Add both KEs and equate to $\frac{3}{4}mu^2$ | **M1** | |
| | Simplify to equation in $\sin\alpha$ | **M1** | |
| | $\sin\alpha = \frac{1}{\sqrt{2}}$, $\alpha = 45°$ | **A1** | |
| | | **4** | |

10 - (9231/33_Summer_2021_Q6)

| | | | |
|---|---|---|---|
| (a) | Let velocities of *A* and *B* along line of centres after collision be $v_1$ and $v_2$. $mv_1 + kmv_2 = mu\cos\theta$ . | **M1** | Momentum, must include *m*, allow cos/sin mix. |
| | $v_2 - v_1 = \frac{1}{3}u\cos\theta$ | **M1** | Restitution, consistent signs, correct way up. |
| | Solve: $v_2 = \frac{4u\cos\theta}{3(1+k)}$ | **A1** | AG shown convincingly. |
| | | **3** | |
| (b) | $v_1 = \frac{(3-k)u\cos\theta}{3(1+k)}$ | **B1** | Or equivalent, may be unsimplified. |
| | Use velocity of *A* with both components. | **B1** | $v_1^2 + (u\sin\theta)^2$ seen. |
| | $\frac{1}{2}kmv_2^2 + \frac{1}{2}m\left(v_1^2 + (u\sin\theta)^2\right) = \frac{3}{10} \times \frac{1}{2}mu^2$ | **M1** | KE after = 30% KE before (all terms present).<br>M0 if incorrect masses. |
| | Substitute from part **(a)** and for $\theta$. | **M1** | Eliminate trigonometric terms, must be KE equation, in terms of $k$ only. |
| | $(3-k)^2 + 16k = 2(1+k)^2$, $k^2 - 6k - 7 = 0$ | **M1** | Obtain simplified quadratic equation in $k$ . |
| | $k = 7$ | **A1** | |
| | | **6** | |

11 - (9231/31_Winter_2021_Q2)

| | | | |
|---|---|---|---|
| (a) | Separate variables and integrate: $\frac{dv}{v} = \left(\frac{1-2t^2}{t}\right)dt$ so $\ln\lvert v\rvert = \ln t - t^2 + c$ | **M1 A1** | Must include logs. Condone missing modulus. |
| | $\lvert v\rvert = Ate^{-t^2}$ , $-v = Ate^{-t^2}$ , $v = -Ate^{-t^2}$ | **A1** | CAO. |
| | | **3** | |
| (b) | $a = \frac{-Ate^{-t^2}\left(1-2t^2\right)}{t} = -Ae^{-t^2}\left(1-2t^2\right)$ | **M1** | Substituting their answer to part **(a)** into given formula |
| | $t=1,\ a=5 \quad (A=5e)$ | **M1** | Use initial condition. |
| | Force = $5me^{1-t^2}\left(2t^2-1\right)$ | **A1** | Use N2L, correct work only. |
| | **Alternative method for (b)** | | |
| | $a = \frac{v\left(1-2t^2\right)}{t}$ substitute $t=1,\ a=5$ so $v=-5$ | **M1** | Use initial condition.<br>Use N2L, correct work only. |
| | Substituting in their answer to part **(a)** so $(A=5e)$ | **M1** | |
| | Force = $5me^{1-t^2}\left(2t^2-1\right)$ | **A1** | |
| | | **3** | |

12 - (9231/31_Winter_2021_Q5)

| | | | |
|---|---|---|---|
| | At $A$: $\uparrow u\sin\theta - 8g \qquad \rightarrow u\cos\theta$ | **M1** | Both. |
| | $\tan\alpha = \dfrac{u\sin\theta - 8g}{u\cos\theta}$ | **A1** | |
| | At $B$: $\uparrow u\sin\theta - 32g \qquad \rightarrow u\cos\theta$ | **M1** | Both. |
| | $\tan\beta = \dfrac{u\sin\theta - 32g}{u\cos\theta}$ | **A1** | |
| | $\dfrac{u\sin\theta - 8g}{u\cos\theta} \times \dfrac{u\sin\theta - 32g}{u\cos\theta} = -1$ | **B1** | Perpendicular directions, so $\tan\alpha \times \tan\beta = -1$. |
| | $u^2 - 320u + 25600 = 0$ | **M1** | Simplify to a quadratic in $u$. |
| | $u = 160$ | **A1** | |
| | | **7** | |

13 - (9231/31_Winter_2021_Q7)

| | | | |
|---|---|---|---|
| (a) | $u\cos\alpha = v\cos\beta$ | **M1** | |
| | $eu\sin\alpha = v\sin\beta$ | **M1** | |
| | Divide: $\tan\beta = e\tan\alpha$ | **A1** | AG. Must see divide OE. |
| | | **3** | |
| (b) | $v\sin\beta = w\cos\gamma \quad (= eu\sin\alpha)$ | **M1** | |
| | $ev\cos\beta = w\sin\gamma \quad (= eu\cos\alpha)$ | **M1** | |
| | Divide: $\tan\gamma = 1/\tan\alpha$ : $\gamma = 90° - \alpha$ | ***A1** | |
| | After second rebound, direction of motion is parallel to initial path. | **DB1** | |
| | | **4** | |
| (c) | Final KE = $\frac{1}{2}m\left((eu\sin\alpha)^2 + (eu\cos\alpha)^2\right) \left(= \frac{1}{2}me^2u^2\right)$ | **M1** | Energy expression in terms of $u$. |
| | So $\frac{1}{2}me^2u^2 = \frac{1}{9}\times\frac{1}{2}mu^2$ giving $e = \frac{1}{3}$ | **A1** | |
| | Part **(a)** gives $\tan(90-\alpha) = e\tan\alpha$ | **M1** | |
| | So $\tan\alpha = \sqrt{3}$ | **A1** | |
| | | **4** | |

14 - (9231/32_Winter_2021_Q1)

| | | | |
|---|---|---|---|
| (a) | Velocity: $\rightarrow u\cos\alpha$ | **B1** | |
| | $\uparrow u\sin\alpha - gT$ | **B1** | Allow 10 for $g$. Must be $T$. |
| | | **2** | |
| (b) | $\dfrac{u\cos\alpha}{u\sin\alpha - gT} = -\dfrac{\sin\alpha}{\cos\alpha}$ oe | **M1 FT** | Allow missing minus sign on RHS for M1.<br>FT from **(a)**. |
| | $T = \dfrac{u}{g\sin\alpha}$ | **A1** | |
| | | **2** | |
| (c) | $\sin\alpha < 1$ giving $T > \dfrac{u}{g}$ | **B1** | AG |
| | | **1** | |

15 - (9231/32_Winter_2021_Q5)

| | | | |
|---|---|---|---|
| (a) | Let speeds of $A$ and $B$ along line of centres after collision be $v_1$ and $v_2$ <br> $mv_1 + \frac{3}{2}mv_2 = -\frac{3}{2}mu\cos 60° + mu \quad \left(= \frac{u}{4}\right)$ | **M1** | Momentum with masses correct. |
| | $v_2 - v_1 = -\frac{2}{3}(-u\cos 60° - u) \quad (= u)$ | **M1** | Restitution, with consistent signs on LHS. |
| | $\left(v_1 = -\frac{1}{2}u\right) \qquad v_2 = \frac{1}{2}u$ | **A1** | |
| | Perpendicular to line of centres, speed of B is $u\sin 60° = \frac{\sqrt{3}}{2}\ u$ | **B1** | |
| | Direction of B is now 60° above line of centres. | **M1** | |
| | Angle of deflection is 60°. | **A1 FT** | FT (120° – *their* direction of B angle) |
| | | **6** | |
| (b) | KE before = $\frac{1}{2}mu^2 + \frac{1}{2}.\frac{3m}{2}u^2 = \frac{5}{4}mu^2$ | **B1** | |
| | KE after = $\frac{1}{2}m\left(\frac{u}{2}\right)^2 + \frac{1}{2}.\frac{3m}{2}\left(\left(\frac{u}{2}\right)^2 + \left(\frac{\sqrt{3}u}{2}\right)^2\right) \quad \left(= \frac{7}{8}mu^2\right)$ | **B1 FT** | FT only *their* speeds from **(a)** |
| | Loss in KE = $\frac{3}{8}mu^2$ | **B1** | |
| | | **3** | |

16 - (9231/32_Winter_2021_Q6)

| | | | |
|---|---|---|---|
| (a) | $2v\dfrac{dv}{dx}=8x-\dfrac{128}{x^3}$ | **M1** | Separate variables and integrate, + $c$ not required for M1. |
| | $v^2=4x^2+64x^{-2}+c$ | **A1** | OE. |
| | $x = 8$, $v = –15$ and $c = –32$ | **M1** | Use initial condition. |
| | $v^2=\dfrac{4}{x^2}\left(x^4-8x^2+16\right)$ or $4x^2-32+\dfrac{64}{x^2}$ | **A1** | Correct expression for $v^2$, AEF. |
| | $v^2=\dfrac{4}{x^2}\left(x^2-4\right)^2$ giving $v=-\dfrac{2}{x}\left(x^2-4\right)$ | **A1** | Convincingly shown, e.g. $v$ is negative initially, AG. |
| | | **5** | |
| (b) | $\dfrac{1}{2}\ln(x^2-4)=-2t\left(+A\right)$ | **M1** | Use $v=\dfrac{dx}{dt}$ and integrate. |
| | $t=0,\ x=8,\ \ A=\dfrac{1}{2}\ln 60$ | **DM1** | Use initial condition. |
| | $\dfrac{1}{2}\ln\left(\dfrac{x^2-4}{60}\right)=-2t$ giving $\dfrac{x^2-4}{60}=e^{-4t}$ | **M1** | Remove log. |
| | $x=\sqrt{4+60e^{-4t}}$ | **A1** | CAO |
| | | **4** | |

17 - (9231/33_Winter_2021_Q2)

| | | | |
|---|---|---|---|
| (a) | Separate variables and integrate: $\frac{dv}{v} = \left(\frac{1-2t^2}{t}\right)dt$ so $\ln\lvert v\rvert = \ln t - t^2 + c$ | **M1 A1** | Must include logs. Condone missing modulus. |
| | $\lvert v\rvert = Ate^{-t^2}$ , $-v = Ate^{-t^2}$ , $v = -Ate^{-t^2}$ | **A1** | CAO. |
| | | **3** | |
| (b) | $a = \frac{-Ate^{-t^2}\left(1-2t^2\right)}{t} = -Ae^{-t^2}\left(1-2t^2\right)$ | **M1** | Substituting their answer to part **(a)** into given formula |
| | $t = 1,\ a = 5 \quad (A = 5e)$ | **M1** | Use initial condition. |
| | Force = $5me^{1-t^2}\left(2t^2-1\right)$ | **A1** | Use N2L, correct work only. |
| | **Alternative method for (b)** | | |
| | $a = \frac{v\left(1-2t^2\right)}{t}$ substitute $t = 1,\ a = 5$ so $v = -5$ | **M1** | Use initial condition.<br>Use N2L, correct work only. |
| | Substituting in their answer to part **(a)** so $(A = 5e)$ | **M1** | |
| | Force = $5me^{1-t^2}\left(2t^2-1\right)$ | **A1** | |
| | | **3** | |

18 - (9231/33_Winter_2021_Q5)

| | | | |
|---|---|---|---|
| | At $A$: $\uparrow u\sin\theta - 8g \quad \rightarrow u\cos\theta$ | **M1** | Both. |
| | $\tan\alpha = \dfrac{u\sin\theta - 8g}{u\cos\theta}$ | **A1** | |
| | At $B$: $\uparrow u\sin\theta - 32g \quad \rightarrow u\cos\theta$ | **M1** | Both. |
| | $\tan\beta = \dfrac{u\sin\theta - 32g}{u\cos\theta}$ | **A1** | |
| | $\dfrac{u\sin\theta - 8g}{u\cos\theta} \times \dfrac{u\sin\theta - 32g}{u\cos\theta} = -1$ | **B1** | Perpendicular directions, so $\tan\alpha \times \tan\beta = -1$. |
| | $u^2 - 320u + 25600 = 0$ | **M1** | Simplify to a quadratic in $u$. |
| | $u = 160$ | **A1** | |
| | | **7** | |

19 - (9231/33_Winter_2021_Q7)

| | | | |
|---|---|---|---|
| (a) | $u\cos\alpha = v\cos\beta$ | **M1** | |
| | $eu\sin\alpha = v\sin\beta$ | **M1** | |
| | Divide: $\tan\beta = e\tan\alpha$ | **A1** | AG. Must see divide OE. |
| | | **3** | |
| (b) | $v\sin\beta = w\cos\gamma \quad (= eu\sin\alpha)$ | **M1** | |
| | $ev\cos\beta = w\sin\gamma \quad (= eu\cos\alpha)$ | **M1** | |
| | Divide: $\tan\gamma = 1/\tan\alpha$ : $\gamma = 90° - \alpha$ | ***A1** | |
| | After second rebound, direction of motion is parallel to initial path. | **DB1** | |
| | | **4** | |
| (c) | Final KE = $\frac{1}{2}m\left((eu\sin\alpha)^2 + (eu\cos\alpha)^2\right) \quad \left(= \frac{1}{2}me^2u^2\right)$ | **M1** | Energy expression in terms of *u*. |
| | So $\frac{1}{2}me^2u^2 = \frac{1}{9}\times\frac{1}{2}mu^2$ giving $e = \frac{1}{3}$ | **A1** | |
| | Part **(a)** gives $\tan(90-\alpha) = e\tan\alpha$ | **M1** | |
| | So $\tan\alpha = \sqrt{3}$ | **A1** | |
| | | **4** | |

20 - (9231/31_Summer_2022_Q6)

| | | | |
|---|---|---|---|
| (a) | Let $v$ and $w$ be speeds after collision:<br>$mv + kmw = mu\cos\alpha$ | **M1** | Momentum along line of centres. |
| | $w - v = \frac{1}{2}u\cos\alpha$ | **M1** | NEL consistent signs. |
| | Add to give $\frac{3u\cos\alpha}{2(1+k)}$ | **A1** | AG<br>Convincing working. |
| | Substitute back or re-solve:<br>$v = \left\lvert\frac{(2-k)u\cos\alpha}{2(1+k)}\right\rvert$ | **A1** | Accept without modulus sign. |
| | | **4** | |

| | | | |
|---|---|---|---|
| (b) | $\sqrt{(u\sin\alpha)^2+\left(\frac{(2-k)u\cos\alpha}{2(1+k)}\right)^2}$ | **B1** | For speed of *A* (SOI). |
| | Equal KE after collision:<br>$\frac{1}{2}km\left(\frac{3u\cos\alpha}{2(1+k)}\right)^2=\frac{1}{2}m\left((u\sin\alpha)^2+\left(\frac{(2-k)u\cos\alpha}{2(1+k)}\right)^2\right)$<br>$\left[9k(\cos\alpha)^2=4(1+k)^2(\sin\alpha)^2+(2-k)^2(\cos\alpha)^2\right]$ | **M1** | Equate KEs. |
| | Use $\tan\alpha=\frac{2}{3}$:<br>$16\left(1+2k+k^2\right)+9\left(4-4k+k^2\right)=81k$ | **M1** | |
| | $25k^2-85k+52=0$ leading to $(5k-4)(5k-13)=0$ | **M1** | Obtain quadratic and attempt to solve. |
| | $k=\frac{4}{5}$ or $\frac{13}{5}$ | **A1** | |
| | | **5** | |

21 - (9231/31_Summer_2022_Q7)

| | | | |
|---|---|---|---|
| (a) | For $Q$: $x = u\cos\beta T$ | **B1** | |
| | For $P$: $x = \frac{35}{2}\cos\alpha\left(T+1\right)$ | **B1** | |
| | Collision, so $\frac{35}{2}\cos\alpha\left(T+1\right) = u\cos\beta T$ | **M1** | Equate and attempt to rearrange. |
| | $\frac{35}{2}\times\frac{3}{5}\left(T+1\right) = u\times\frac{2}{\sqrt{5}}T$<br>$4uT = 21\sqrt{5}\left(T+1\right)$ | **A1** | AG<br>Shown convincingly. |
| | | **4** | |
| (b) | Vertical motion to collision:<br>For $Q$: $y = u\sin\beta T - \frac{1}{2}gT^2$<br>For $P$: $y = \frac{35}{2}\sin\alpha\left(T+1\right) - \frac{1}{2}g(T+1)^2$ | **M1 A1** | M1 for both expressions, one correct. |
| | Equate: $u\times\frac{1}{\sqrt{5}}T - \frac{1}{2}gT^2 = \frac{35}{2}\times\frac{4}{5}\left(T+1\right) - \frac{1}{2}g(T+1)^2$<br>$14\left(T+1\right) - \frac{1}{2}g\left(T^2+2T+1-T^2\right) = \frac{21}{4}\left(T+1\right)$ | **M1** | Equate and attempt to solve |
| | $16T+36 = 21T+21, \quad 15 = 5T$<br>$T = 3$ | **A1** | |
| | | **4** | |

| | | | |
|---|---|---|---|
| (c) | $x = 42$ | **B1** | |
| | $\lvert y \rvert = 24$ | **M1** | |
| | $y = -24$ (or 24 m below $O$) | **A1** | Correct sign or in words. |
| | | **3** | |

22 - (9231/32_Summer_2022_Q6)

| | | | |
|---|---|---|---|
| (a) | Let $v$ and $w$ be speeds after collision:<br>$mv + kmw = mu\cos\alpha$ | **M1** | Momentum along line of centres. |
| | $w - v = \frac{1}{2}u\cos\alpha$ | **M1** | NEL consistent signs. |
| | Add to give $\frac{3u\cos\alpha}{2(1+k)}$ | **A1** | AG<br>Convincing working. |
| | Substitute back or re-solve:<br>$v = \left\lvert \frac{(2-k)u\cos\alpha}{2(1+k)} \right\rvert$ | **A1** | Accept without modulus sign. |
| | | **4** | |

| | | | |
|---|---|---|---|
| (b) | $\sqrt{(u\sin\alpha)^2+\left(\frac{(2-k)u\cos\alpha}{2(1+k)}\right)^2}$ | **B1** | For speed of *A* (SOI). |
| | Equal KE after collision: $\frac{1}{2}km\left(\frac{3u\cos\alpha}{2(1+k)}\right)^2=\frac{1}{2}m\left((u\sin\alpha)^2+\left(\frac{(2-k)u\cos\alpha}{2(1+k)}\right)^2\right)$ $\left[9k(\cos\alpha)^2=4(1+k)^2(\sin\alpha)^2+(2-k)^2(\cos\alpha)^2\right]$ | **M1** | Equate KEs. |
| | Use $\tan\alpha=\frac{2}{3}$: $16\left(1+2k+k^2\right)+9\left(4-4k+k^2\right)=81k$ | **M1** | |
| | $25k^2-85k+52=0$ leading to $(5k-4)(5k-13)=0$ | **M1** | Obtain quadratic and attempt to solve. |
| | $k=\frac{4}{5}$ or $\frac{13}{5}$ | **A1** | |
| | | **5** | |

23 - (9231/32_Summer_2022_Q7)

| | | | |
|---|---|---|---|
| (a) | For $Q$: $x = u\cos\beta T$ | **B1** | |
| | For $P$: $x = \frac{35}{2}\cos\alpha\left(T+1\right)$ | **B1** | |
| | Collision, so $\frac{35}{2}\cos\alpha\left(T+1\right) = u\cos\beta T$ | **M1** | Equate and attempt to rearrange. |
| | $\frac{35}{2}\times\frac{3}{5}\left(T+1\right) = u\times\frac{2}{\sqrt{5}}T$<br>$4uT = 21\sqrt{5}\left(T+1\right)$ | **A1** | AG<br>Shown convincingly. |
| | | **4** | |
| (b) | Vertical motion to collision:<br>For $Q$: $y = u\sin\beta T - \frac{1}{2}gT^2$<br>For $P$: $y = \frac{35}{2}\sin\alpha\left(T+1\right) - \frac{1}{2}g(T+1)^2$ | **M1 A1** | M1 for both expressions, one correct. |
| | Equate: $u\times\frac{1}{\sqrt{5}}T - \frac{1}{2}gT^2 = \frac{35}{2}\times\frac{4}{5}\left(T+1\right) - \frac{1}{2}g(T+1)^2$<br>$14\left(T+1\right) - \frac{1}{2}g\left(T^2+2T+1-T^2\right) = \frac{21}{4}\left(T+1\right)$ | **M1** | Equate and attempt to solve |
| | $16T+36 = 21T+21, \quad 15 = 5T$<br>$T = 3$ | **A1** | |
| | | **4** | |

| | | | |
|---|---|---|---|
| (c) | $x = 42$ | **B1** | |
| | $\lvert y \rvert = 24$ | **M1** | |
| | $y = -24$ (or 24 m below $O$) | **A1** | Correct sign or in words. |
| | | **3** | |

24 - (9231/33_Summer_2022_Q6)

| | | | |
|---|---|---|---|
| (a) | Let $v$ be speed of rebound from 1st collision:<br>Energy loss: $\frac{1}{2}mv^2 = \frac{2}{5} \times \frac{1}{2}mu^2, \quad v^2 = \frac{2}{5}u^2$ | **B1** | Energy loss. |
| | $v\cos\alpha = u\cos\theta$<br>$v\sin\alpha = eu\sin\theta$ | **B1** | Both. |
| | Combine to form equation in $e$ only<br>$\frac{2}{5} = \frac{1}{5} + e^2 \times \frac{4}{5}$ | **M1** | $v^2 = (u\cos\theta)^2 + (eu\sin\theta)^2$ |
| | $e = \frac{1}{2}$ | **A1** | |
| | | **4** | |

| | | | |
|---|---|---|---|
| (b) | $\tan\alpha = e\tan\theta$, so $\tan\alpha = 1$, $\alpha = 45°$ | **B1** | |
| | For 2nd collision<br>$w\cos\beta = v\cos(180-60-\alpha)$<br>$w\sin\beta = ev\sin(180-60-\alpha)$ | **M1** | Both.<br>May be implied by the A1. |
| | $\tan\beta = e\tan(120 - their\ \alpha)$ | **M1** | Divide to find $\beta$. |
| | $\beta = 61.8°$ | **A1** | |
| | | **4** | |

25 - (9231/31_Winter_2022_Q4)

| | | | |
|---|---|---|---|
| (a) | $m\frac{dv}{dt} = 50 - 2v^2 \quad \frac{dv}{dt} = 4\left(25 - v^2\right)$ | **B1** | N2L |
| | $\frac{1}{10}\int \frac{1}{5-v} + \frac{1}{5+v}\,dv = \int 4dt$ | **M1** | Separate variables and use partial fractions. |
| | $\frac{1}{10}\left(-\ln(5-v) + \ln(5+v)\right) = 4t + A$ | **M1 A1** | Integrate into log terms.<br>(Note: formula on MF19). |
| | Use $t = 0, v = 3$ to give $A = \frac{1}{10}\ln 4$ | **M1** | Use initial condition. |
| | $4t = \frac{1}{10}\ln\frac{5+v}{4(5-v)}$ leading to $\frac{5+v}{20-4v} = e^{40t}$ | **M1** | Rearrange to make $v$ the subject. |
| | $v = \frac{5\left(4 - e^{-40t}\right)}{4 + e^{-40t}}$ | **A1** | |
| | | **7** | |
| (b) | As $t \to \infty, v \to 5$ | **B1** | |
| | | **1** | |

26 - (9231/31_Winter_2022_Q6)

| | | | |
|---|---|---|---|
| (a) | Let speed of A after collision be $\rightarrow v_A$ and speed of B perpendicular to line of centres be $\downarrow v$<br>Along line of centres: $mu - km\frac{5}{8}u\cos\alpha = mv_A$ | **M1** | Momentum. |
| | NEL: $0 - v_A = e\left(\frac{5}{8}u\ \cos\alpha + u\right)$ | **M1** | NEL |
| | So $u - \frac{5}{8}ku\cos\alpha = -\frac{2}{3}\left(\frac{5}{8}u\ \cos\alpha + u\right)$ | **M1** | Solve. |
| | Substitute for cos, to give $k = 4$ | **A1** | |
| | | **4** | |
| (b) | $v_B = \frac{5}{8}u\sin\alpha\ = \frac{3}{8}u$ | **B1** | Velocity perpendicular to line of centres |
| | $v_A = -u$ | **B1 FT** | |
| | KE before = $\frac{1}{2}mu^2 + \frac{1}{2}km\left(\frac{5}{8}u\right)^2 = \frac{1}{2}mu^2 + \frac{25}{32}mu^2 = \frac{41}{32}mu^2$<br>KE after = $\frac{1}{2}m{v_A}^2 + \frac{1}{2}km{v_B}^2 = \frac{1}{2}mu^2 + 2m\frac{9}{64}u^2 = \frac{25}{32}mu^2$ | **M1** | NOTE: KE before and after for $A$ is unchanged.<br>Both. |
| | Loss = $mu^2\left(\frac{41}{32} - \frac{25}{32}\right) = \frac{1}{2}mu^2$ | **A1** | |
| | | **4** | |

27 - (9231/32_Winter_2022_Q3)

| | | | |
|---|---|---|---|
| | In equilibrium, $\dfrac{\frac{16}{3}Mge}{a} = 4Mg, \quad e = \dfrac{3}{4}a$ | **B1** | |
| | In subsequent motion,<br>Loss in GPE = gain in EPE + gain in KE | **M1** | Energy equation with GPE and KE terms correct and at least one EPE term.<br>Dimensionally correct. |
| | $\dfrac{6Mga}{4} = \dfrac{1}{2}.\dfrac{16}{3}.\dfrac{Mg}{a}.\left(a^2 - \left(\dfrac{3a}{4}\right)^2\right) + \dfrac{1}{2}.6Mv^2$ | **B1** | EPE correct. |
| | | **A1** | All correct. |
| | $\dfrac{3Mga}{2} = \dfrac{8Mg}{3a}.\dfrac{7}{16}a^2 + 3Mv^2$ etc | **M1** | Attempt to find $v$ in terms of $a$ and $g$. |
| | $\dfrac{ga}{3} = 3v^2, \quad v = \dfrac{1}{3}\sqrt{ga}$ | **A1** | |
| | | **6** | |

28 - (9231/32_Winter_2022_Q7)

| | | | |
|---|---|---|---|
| (a) | Let $v, w$ be speeds of $A$ and $B$ along line of centres after collision $mv+\frac{1}{2}mw=mu\cos\alpha-\frac{1}{2}m.2u\cos\beta$ | **M1** | Momentum: masses correct, opposite signs on RHS. |
| | $w-v=e(2u\cos\beta+u\cos\alpha)$ | **M1** | NEL: LHS signs must be consistent with momentum equation, same sign for both terms on RHS. |
| | $\alpha+\beta=90°$, so $\cos\beta=\sin\alpha$ Use this fact and solve to find $w$ | **M1** | Solve to find an expression of the correct form. |
| | $w=\frac{2}{3}u\left(\frac{1}{4}\sin\alpha+\frac{13}{8}\cos\alpha\right)$ | **A1** | |
| | | **4** | |
| (b) | Perpendicular to line of centres, speed of $B$ is $2u\sin\beta=2u\cos\alpha$ | **B1** | |
| | After, velocity of $B$ makes angle $\alpha$ with line of centres, so $\tan\alpha=\frac{2u\cos\alpha}{w}$ | **B1** | |
| | $\frac{\sin\alpha}{\cos\alpha}=\frac{2u\cos\alpha}{\frac{2}{3}u\left(\frac{1}{4}\sin\alpha+\frac{13}{8}\cos\alpha\right)}$ giving | **M1*** | Obtain homogeneous equation in cos and sin or an equation in tan |
| | $3(\cos\alpha)^2=\frac{1}{4}(\sin\alpha)^2+\frac{13}{8}\sin\alpha\cos\alpha$ <br> $2(\tan\alpha)^2+13\tan\alpha-24=0$, $(2\tan\alpha-3)(\tan\alpha+8)=0$ | **DM1** | Obtain quadratic and solve to find values of $\tan\alpha$ |
| | $\tan\alpha=\frac{3}{2}$ | **A1** | |
| | | **5** | |

29 - (9231/33_Winter_2022_Q6)

| | | | |
|---|---|---|---|
| (a) | Let speed of A after collision be $\rightarrow v_A$ and speed of B perpendicular to line of centres be $\downarrow v$<br>Along line of centres: $mu - km\frac{5}{8}u\cos\alpha = mv_A$ | **M1** | Momentum. |
| | NEL: $0 - v_A = e\left(\frac{5}{8}u\ \cos\alpha + u\right)$ | **M1** | NEL |
| | So $u - \frac{5}{8}ku\cos\alpha = -\frac{2}{3}\left(\frac{5}{8}u\ \cos\alpha + u\right)$ | **M1** | Solve. |
| | Substitute for cos, to give $k = 4$ | **A1** | |
| | | **4** | |
| (b) | $v_B = \frac{5}{8}u\sin\alpha\ = \frac{3}{8}u$ | **B1** | Velocity perpendicular to line of centres |
| | $v_A = -u$ | **B1 FT** | |
| | KE before = $\frac{1}{2}mu^2 + \frac{1}{2}km\left(\frac{5}{8}u\right)^2 = \frac{1}{2}mu^2 + \frac{25}{32}mu^2 = \frac{41}{32}mu^2$<br>KE after = $\frac{1}{2}mv_A{}^2 + \frac{1}{2}kmv_B{}^2 = \frac{1}{2}mu^2 + 2m\frac{9}{64}u^2 = \frac{25}{32}mu^2$ | **M1** | NOTE: KE before and after for *A* is unchanged.<br>Both. |
| | Loss = $mu^2\left(\frac{41}{32} - \frac{25}{32}\right) = \frac{1}{2}mu^2$ | **A1** | |
| | | **4** | |

30 - (9231/31_Summer_2023_Q2)

| | | | |
|---|---|---|---|
| | Parallel to wall $v\cos\theta = u\cos\alpha$<br>Perpendicular to wall $v\sin\theta = eu\sin\alpha$ | **M1** | Both |
| | Dividing, $e = \dfrac{1}{2\tan\alpha}$ | **A1** | AEF |
| | KE reduced by 20%, so $\frac{1}{2}mu^2\left(\cos^2\alpha + e^2\sin^2\alpha\right) = \frac{4}{5}\times\frac{1}{2}mu^2$ | **M1** | Dimensionally correct equation in $u$ or $v$, but not both. Must have either $\alpha$ or $\theta$, but not both.<br>Must see $\frac{4}{5}$ on the correct side of the equation. |
| | Eliminate $e$: $\cos\alpha = \dfrac{4}{5}$ | **A1** | |
| | $e = \dfrac{2}{3}$ | **A1** | |
| | **Alternative method** | | |
| | Parallel to wall $v\cos\theta = u\cos\alpha$<br>Perpendicular to wall $v\sin\theta = eu\sin\alpha$ | **M1** | Both |
| | $\left[\sin(\theta) = \dfrac{\sqrt{5}}{5},\ \cos(\theta) = \dfrac{2\sqrt{5}}{5}\right]$ $u\sin(\alpha) = \dfrac{\sqrt{5}v}{5e},\ u\cos(\alpha) = \dfrac{2\sqrt{5}v}{5}$ | **A1** | |
| | $u^2 = \left[u^2cos^2(\alpha) + u^2sin^2(\alpha)\right] = \dfrac{4v^2}{5} + \dfrac{v^2}{5e^2}$ | **A1** | AEF, e.g. $\dfrac{v^2}{5}\left(4 + \dfrac{1}{e^2}\right)$ |

| | | | |
|---|---|---|---|
| | $\frac{1}{2}mv^2 = \left[\frac{4}{5}\times\frac{1}{2}mu^2 = \right]\frac{2}{5}m\frac{v^2}{5}\left(4+\frac{1}{e^2}\right)$ | **M1** | Dimensionally correct equation in $v$.<br>Must have either $\alpha$ or $\theta$, but not both.<br>Must see $\frac{4}{5}$ or $\frac{2}{5}$ on the correct side of the equation. |
| | $e = \frac{2}{3}$ | **A1** | |
| | | **5** | |

31 - (9231/32_Summer_2023_Q2)

| | | | |
|---|---|---|---|
| | Parallel to wall $v\cos\theta = u\cos\alpha$<br>Perpendicular to wall $v\sin\theta = eu\sin\alpha$ | **M1** | Both |
| | Dividing, $e = \frac{1}{2\tan\alpha}$ | **A1** | AEF |
| | KE reduced by 20%, so $\frac{1}{2}mu^2\left(\cos^2\alpha + e^2\sin^2\alpha\right) = \frac{4}{5}\times\frac{1}{2}mu^2$ | **M1** | Dimensionally correct equation in $u$ or $v$, but not both. Must have either $\alpha$ or $\theta$, but not both.<br>Must see $\frac{4}{5}$ on the correct side of the equation. |
| | Eliminate $e$: $\cos\alpha = \frac{4}{5}$ | **A1** | |
| | $e = \frac{2}{3}$ | **A1** | |

| | | | |
|---|---|---|---|
| | **Alternative method** | | |
| | Parallel to wall $v\cos\theta = u\cos\alpha$<br>Perpendicular to wall $v\sin\theta = eu\sin\alpha$ | **M1** | Both |
| | $\left[\sin(\theta)=\frac{\sqrt{5}}{5},\ \cos(\theta)=\frac{2\sqrt{5}}{5}\right]$ $u\sin(\alpha)=\frac{\sqrt{5}v}{5e},\ u\cos(\alpha)=\frac{2\sqrt{5}v}{5}$ | **A1** | |
| | $u^2=\left[u^2cos^2(\alpha)+u^2sin^2(\alpha)\right]=\frac{4v^2}{5}+\frac{v^2}{5e^2}$ | **A1** | AEF, e.g. $\frac{v^2}{5}\left(4+\frac{1}{e^2}\right)$ |
| | $\frac{1}{2}mv^2=\left[\frac{4}{5}\times\frac{1}{2}mu^2=\right]\frac{2}{5}m\frac{v^2}{5}\left(4+\frac{1}{e^2}\right)$ | **M1** | Dimensionally correct equation in $v$.<br>Must have either $\alpha$ or $\theta$, but not both.<br>Must see $\frac{4}{5}$ or $\frac{2}{5}$ on the correct side of the equation. |
| | $e=\frac{2}{3}$ | **A1** | |
| | | **5** | |

32 - (9231/33_Summer_2023_Q4)

| | | | |
|---|---|---|---|
| (a) | Let speeds of $A$ and $B$ along line of centres after collision be $V_A$ and $V_B$<br>$V_A + V_B = u\cos 30°$ (1) | **M1** | Allow sign errors, allow missing $m$. |
| | $-V_A + V_B = eu\cos 30°$ (2) | **M1** | Signs on LHS must be consistent with (1). |
| | Speeds perpendicular to line of centres after collision are $u\sin 30°$ and $2u$<br>Moving in same direction, so $\frac{V_A}{u\sin 30°} = \frac{V_B}{2u}$ (3) | **B1** | SOI<br>$V_B = 4V_A$ |
| | Use $V_B = 4V_A$ in (1): $5V_A = u\cos 30°$<br>From (2): $3V_A = eu\cos 30°$ then<br>Combine to find equation in $e$ only. | **M1** | A complete method to find equation in $e$ only |
| | $e = \frac{3}{5}$ | **A1** | |
| (a) | **Alternative method for (a)** | | |
| | Let speeds of $A$ and $B$ along line of centres after collision be $V_A$ and $V_B$<br>$V_A + V_B = u\cos 30°$ (1) | **M1** | Allow sign errors, allow missing $m$. |
| | $-V_A + V_B = eu\cos 30°$ (2) | **M1** | Signs on LHS must be consistent with (1). |
| | Speeds perpendicular to line of centres after collision are $u\sin 30°$ and $2u$<br>Moving in same direction, so $\frac{V_A}{u\sin 30°} = \frac{V_B}{2u}$ (3) | **B1** | SOI<br>$V_B = 4V_A$ |

| | | | |
|---|---|---|---|
| (a) | Solve (1) and (2): $V_A = \frac{1}{2}u(1-e)\cos 30°$ , $V_B = \frac{1}{2}u(1+e)\cos 30°$<br>Substitute in (3) to find equation in $e$ only . | **M1** | Note: $V_A = \frac{u}{10}\sqrt{3},\ V_B = \frac{4u}{10}\sqrt{3}$ |
| | $e = \frac{3}{5}$ | **A1** | |
| | | **5** | |
| (b) | KE after = $\frac{1}{2}m\left(V_A^2 + \left(\frac{u}{2}\right)^2\right) + \frac{1}{2}m\ ((2u)^2 + V_B^2\ )$ | **B1** | Correct expression for KE for one of the spheres, after collision, with both components. |
| | KE for A after = $\frac{7}{50}mu^2$ or KE for B after = $\frac{56}{25}mu^2$<br>or KE loss for A = $\frac{9}{25}mu^2$ or KE **gain** for B = $\frac{6}{25}mu^2$ | **B1** | Implied by total KE after = $\frac{119}{50}mu^2$ . |
| | Total loss in KE = $\frac{3}{25}mu^2$ | **B1** | Term $\frac{1}{2}m(2u)^2$ may be omitted from KE of $B$ before and after. |
| | | **3** | |