A LEVEL Cambridge Topical Past Papers

FURTHER MATHEMATICS

9231 P1,P2

2020 — 2025

EXAMINENT
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1 - (9231/11_Summer_2020_Q2)

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

2 - (9231/12_Summer_2020_Q2)

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

3 - (9231/13_Summer_2020_Q1)

The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

4 - (9231/11_Winter_2020_Q3)

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

5 - (9231/12_Winter_2020_Q1)

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

6 - (9231/13_Winter_2020_Q3)

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

7 - (9231/11_Summer_2021_Q3)

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

8 - (9231/12_Summer_2021_Q3)

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

9 - (9231/13_Summer_2021_Q2)

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. [3]

(d) Find the value of $\frac{1}{\alpha + \beta} + \frac{1}{\beta + \gamma} + \frac{1}{\gamma + \alpha}$. [2]

10 - (9231/11_Winter_2021_Q1)

It is given that

$$\alpha+\beta+\gamma=3, \quad \alpha^2+\beta^2+\gamma^2=5, \quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

11 - (9231/12_Winter_2021_Q4)

The cubic equation $x^3+2x^2+3x+3=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

(b) Show that $\alpha^3+\beta^3+\gamma^3=1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

12 - (9231/13_Winter_2021_Q1)

It is given that

$$\alpha+\beta+\gamma=3, \quad \alpha^2+\beta^2+\gamma^2=5, \quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

13 - (9231/11_Summer_2022_Q4)

The cubic equation $2x^3+5x^2-6=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6}+\frac{1}{\beta^6}+\frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

14 - (9231/12_Summer_2022_Q4)

The cubic equation $2x^3+5x^2-6=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6}+\frac{1}{\beta^6}+\frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

15 - (9231/11_Winter_2022_Q1)

The cubic equation $x^3+bx^2+d=0$ has roots α, β, γ, where $\alpha=\beta$ and $d\neq 0$.

(a) Show that $4b^3+27d=0$. [5]

(b) Given that $2\alpha^2+\gamma^2=3b$, find the values of b and d. [3]

16 - (9231/12_Winter_2022_Q2)

The equation $x^4+3x^2+2x+6=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}$. [2]

17 - (9231/13_Winter_2022_Q1)

The cubic equation $x^3+bx^2+d=0$ has roots α, β, γ, where $\alpha=\beta$ and $d \neq 0$.

(a) Show that $4b^3+27d=0$. [5]

(b) Given that $2\alpha^2+\gamma^2=3b$, find the values of b and d. [3]

18 - (9231/11_Summer_2023_Q2)

The cubic equation $x^3+4x^2+6x+1=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2\right)=n(n^2+an+b),$$

where a and b are constants to be determined. [6]

19 - (9231/12_Summer_2023_Q2)

The cubic equation $x^3+4x^2+6x+1=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2\right)=n(n^2+an+b),$$

where a and b are constants to be determined. [6]

20 - (9231/13_Summer_2023_Q3)

The equation $x^4-x^2+2x+5=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2+\beta^2+\gamma^2+\delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

21 - (9231/13_Summer_2023_Q7)

The curve C has equation $y=\frac{x^2+2x+1}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the turning points on C. [3]

(c) Sketch C. [3]

(d) Sketch the curves with equations $y=\left|\frac{x^2+2x+1}{x-3}\right|$ and $y^2=\frac{x^2+2x+1}{x-3}$ on a single diagram, clearly identifying each curve. [4]

22 - (9231/11_Winter_2023_Q3)

The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

23 - (9231/12_Winter_2023_Q4)

The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

The sum $(3\alpha + 1)^n + (3\beta + 1)^n + (3\gamma + 1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

(c) Find the values of S_{-1} and S_{-2} . [3]

24 - (9231/13_Winter_2023_Q3)

The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

25 - (9231/11_Summer_2024_Q1)

The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

(c) Deduce a cubic equation whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$. [1]

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

26 - (9231/12_Summer_2024_Q1)

The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

(c) Deduce a cubic equation whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$. [1]

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

27 - (9231/13_Summer_2024_Q2)

The cubic equation $x^3 + 2x^2 + 3x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2 + 1, \beta^2 + 1, \gamma^2 + 1$. [3]

(b) Find the value of $(\alpha^2 + 1)^2 + (\beta^2 + 1)^2 + (\gamma^2 + 1)^2$. [2]

(c) Find the value of $(\alpha^2 + 1)^3 + (\beta^2 + 1)^3 + (\gamma^2 + 1)^3$. [2]

28 - (9231/11_Winter_2024_Q3)

The quartic equation $x^4 + 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ and state the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [5]

(b) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [3]

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. [2]

29 - (9231/12_Winter_2024_Q3)

It is given that

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 2, \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= 3, \\ \alpha^3 + \beta^3 + \gamma^3 + \delta^3 &= 4. \end{aligned}$$

(a) Find the value of $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$. [2]

(b) Find the value of $\alpha^2\beta + \alpha^2\gamma + \alpha^2\delta + \beta^2\alpha + \beta^2\gamma + \beta^2\delta + \gamma^2\alpha + \gamma^2\beta + \gamma^2\delta + \delta^2\alpha + \delta^2\beta + \delta^2\gamma$. [3]

(c) It is given that α, β, γ, δ are the roots of the equation

$$6x^4 - 12x^3 + 3x^2 + 2x + 6 = 0.$$

(i) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [3]

(ii) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [2]

30 - (9231/13_Winter_2024_Q3)

The quartic equation $x^4 + 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ and state the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [5]

(b) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [3]

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. [2]

31 - (9231/21_Winter_2024_Q4)

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue. [2]

(b) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the other eigenvalues of $\mathbf{A}$. [3]

(c) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

32 - (9231/22_Winter_2024_Q4)

(a) Use de Moivre's theorem to show that

$$\cot 6\theta = \frac{\cot^6\theta - 15\cot^4\theta + 15\cot^2\theta - 1}{6\cot^5\theta - 20\cot^3\theta + 6\cot\theta}.$$ [6]

(b) Hence obtain the roots of the equation

$$x^6 - 6x^5 - 15x^4 + 20x^3 + 15x^2 - 6x - 1 = 0$$

in the form $\cot(q\pi)$, where q is a rational number. [4]

33 - (9231/23_Winter_2024_Q1)

Find the set of values of k for which the system of equations

$$\begin{aligned} x + 5y + 6z &= 1, \\ kx + 2y + 2z &= 2, \\ -3x + 4y + 8z &= 3, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

34- (9231/23_Winter_2024_Q4)

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue. [2]

(b) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the other eigenvalues of $\mathbf{A}$. [3]

(c) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

35 - (9231/11_Summer_2025_Q2)

The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1,\ \beta^3 - 1,\ \gamma^3 - 1$. [3]

(b) Find the value of $(\alpha^3 - 1)^2 + (\beta^3 - 1)^2 + (\gamma^3 - 1)^2$. [2]

(c) Find the value of $(\alpha^3 - 1)^3 + (\beta^3 - 1)^3 + (\gamma^3 - 1)^3$. [2]

36- (9231/12_Summer_2025_Q2)

The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1,\ \beta^3 - 1,\ \gamma^3 - 1$. [3]

(b) Find the value of $(\alpha^3 - 1)^2 + (\beta^3 - 1)^2 + (\gamma^3 - 1)^2$. [2]

(c) Find the value of $(\alpha^3 - 1)^3 + (\beta^3 - 1)^3 + (\gamma^3 - 1)^3$. [2]

37 - (9231/13_Summer_2025_Q3)

The quartic equation $x^4 + 7x^2 + 3x + 22 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [2]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

(c) Use standard results from the list of formulae (MF19) to find the value of

$$\sum_{r=1}^{10}\left(\left(\alpha^2 + r\right)^2 + \left(\beta^2 + r\right)^2 + \left(\gamma^2 + r\right)^2 + \left(\delta^2 + r\right)^2\right).$$ [5]

38- (9231/14_Summer_2025_Q4)

The cubic equation $x^3 + bx^2 + cx - 1 = 0$, where b and c are constants, has roots α, β, γ.

It is given that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 = 3$. [4]

(b) It is given that $\alpha^3 + \beta^3 + \gamma^3 = 3$ and that the constants b and c are positive.

Find the values of b and c. [6]

39- (9231/21_Summer_2025_Q1)

Find the roots of the equation $z^3 = 27 - 27\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

40- (9231/23_Summer_2025_Q5)

(a) Use de Moivre's theorem to show that

$$\sec 5\theta = \frac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}.$$ [6]

(b) Hence, obtain the roots of the equation

$$\sqrt{3}x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\sec(q\pi)$, where q is rational. [4]

41- (9231/24_Summer_2025_Q5)

(a) Use de Moivre's theorem to show that

$$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta. \qquad [5]$$

(b) Hence find all roots of the equation

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$

in the form $\sin q\pi$, where q is a rational number. [3]

42 - (9231/11_Winter_2025_Q4)

The quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Show that a quartic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1, 2\delta + 1$ is

$$y^4 - 2y^3 + 4y^2 + 2y + 11 = 0. \qquad [4]$$

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n + (2\delta + 1)^n$ is denoted by S_n.

(b) Find the value of S_2. [2]

(c) Given that $S_3 = -22$, find the value of S_4. [2]

43 - (9231/12_Winter_2025_Q2)

The cubic equation $x \quad + bx \quad + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 3,$$
$$\alpha^4 + \beta^4 + \gamma^4 = 5.$$

(a) Find the values of b and c. [3]

(b) Find the value of d. [5]

44 - (9231/14_Winter_2025_Q2)

The cubic equation $x^3 + bx^2 + cx + d = 0$, where $d \neq 0$, has roots α, β, γ such that $\gamma = \dfrac{1}{\beta}$.

(a) Show that $\beta + \dfrac{1}{\beta} = d - b$. [3]

(b) Show also that $\beta + \dfrac{1}{\beta} = \dfrac{1-c}{d}$. [2]

(c) It is given that $b = 3$, $c = -3$ and $d > 0$.

(i) Find the value of d. [2]

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

1 - (9231/11_Summer_2020_Q2)

(a)	$y = x^2$	**B1**
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py+5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	**M1**
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	**A1**
		3
(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \dfrac{p^2+36}{36}$	**B1**
	$\dfrac{p^2+36}{36} = -\dfrac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	**M1**
	$p = -6$	**A1**
		3
(b)(ii)	$6(\alpha^3+\beta^3+\gamma^3) = 6(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 15$	**M1**
	$\alpha^3+\beta^3+\gamma^3 = 5$	**A1**
		2

2 - (9231/12_Summer_2020_Q2)

(a)	$y = x^2$	**B1**
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}\left(6y - 3\right) = -py + 5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	**M1**
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	**A1**
		3
(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2+36}{36}$	**B1**
	$\frac{p^2+36}{36} = -\frac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	**M1**
	$p = -6$	**A1**
		3
(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	**M1**
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	**A1**
		2

3 - (9231/13_Summer_2020_Q1)

(a)	$y = x^{-1}$	**B1**
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	**M1 A1**
		3
(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	**M1 A1**
(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	**M1 A1**
		4

4 - (9231/11_Winter_2020_Q3)

(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	**M1**	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	**A1**	
		3	
(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \;\; \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	**M1**	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	**A1**	AG
		3	
(c)	$\alpha^3\beta^3\gamma^3 = -1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	**M1 A1**	Evaluates determinant.
	$2c^3 - 4 = 0$	**M1**	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	**A1**	
		5	

5 - (9231/12_Winter_2020_Q1)

(a)	$d = 1$	**B1**	
		1	
(b)	$y = x + 1 \Rightarrow x = y - 1$	**B1**	Uses correct substitution.
	$y^3 + (b-3)y^2 + (c-2b+3)y + b - c = 0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\beta+1)(\gamma+1) = c - 2b + 3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1) = 3 - b,\ \ (\alpha+1)(\beta+1)(\gamma+1) = c - b$	**M1**	Using these relationships.
	$y^3 + (b-3)y^2 + (c-2b+3)y + b - c = 0)$	**A1**	
		3	
(c)	$\beta + 1 = -(b-3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1) = c - 2b + 3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1) = -(b-c)$	**M1**	Applies product of roots.
	$\Rightarrow (c-2b+3)(b-3) = b - c$	**A1**	AG
		4	

6 - (9231/13_Winter_2020_Q3)

(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	**M1**	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	**A1**	
		3	
(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	**M1**	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	**A1**	AG
		3	
(c)	$\alpha^3\beta^3\gamma^3 = -1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	**M1 A1**	Evaluates determinant.
	$2c^3 - 4 = 0$	**M1**	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	**A1**	
		5	

7 - (9231/11_Summer_2021_Q3)

(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	
(b)	$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 4$	**M1**	Uses original equation.
	$= 20$	**A1**	
		2	

8 - (9231/12_Summer_2021_Q3)

(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	
(b)	$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 4$	**M1**	Uses original equation.
	$= 20$	**A1**	
		2	

9 - (9231/13_Summer_2021_Q2)

(a)	$S_1 = 2$	**B1**	
	$S_2 = S_1^{\ 2} - 2(0)$	**M1**	Uses formula for sum of squares.
	$= 4$	**A1**	Correct answer implies M1A1.
		3	
(b)(i)	$S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n$	**B1**	CAO or as a single fraction.
		1	
(b)(ii)	$S_4 = 2S_3 - \frac{3}{2}S_1 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$	**M1**	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3 = \frac{7}{2}\right]$.
	$= 4$	**A1**	
		2	
(c)	$x = 2 - y$	**B1**	SOI
	$2(2-y)^3 - 4(2-y)^2 + 3 = 0$	**M1**	Makes *their* substitution.
	$2y^3 - 8y^2 + 8y - 3 = 0$	**A1**	OE but must be an equation.
		3	

(d)	$\dfrac{\frac{8}{2}}{\frac{-3}{2}}$ OR Or use $2S_2 - 8S_1 + 8S_0 - 3S_{-1} = 0$ with substitution of their values	**M1**	Uses $\dfrac{1}{\alpha'} + \dfrac{1}{\beta'} + \dfrac{1}{\gamma'} = \dfrac{\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'}{\alpha'\beta'\gamma'}$.
	$= \dfrac{8}{3}$	**A1 FT**	FT from **2(c).**
		2	

10 - (9231/11_Winter_2021_Q1)

	$b = -(\alpha + \beta + \gamma) = -3$	**B1**	
	$5 = 3^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$	**M1 A1**	Uses formula for sum of squares.
	$c = 2$	**A1**	
	$6 - 3(5) + 2(3) + 3d = 0$	**M1**	Uses original equation or formula for sum of cubes.
	[Equation is $x^3 - 3x^2 + 2x + 1 = 0$] $d = 1$	**A1**	
		6	

11 - (9231/12_Winter_2021_Q4)

(a)	$(-2)^2 - 2(3)$	**M1**	Uses formula for sum of squares.
	-2	**A1**	
		2	
(b)	$\alpha^3+\beta^3+\gamma^3 = -2(-2)-3(-2)-3(3)$	**M1**	Uses original equation or formula for sum of cubes.
	1	**A1**	AG
		2	
(c)	$(\alpha+r)^3 = \alpha^3 + 3\alpha^2 r + 3\alpha r^2 + r^3$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right) = \sum_{r=1}^{n}\left(1+3(-2)r+3(-2)r^2+3r^3\right)$	**M1 A1**	Collects like terms and uses results from parts **(a)** and **(b)**.
	$n-6\left(\frac{1}{2}n(n+1)\right)-6\left(\frac{1}{6}n(n+1)(2n+1)\right)+\frac{3}{4}n^2(n+1)^2$ $n-3n(n+1)-n(n+1)(2n+1)+\frac{3}{4}n^2(n+1)^2$	**M1**	Applies formulae from MF19.
	$n+\frac{1}{4}n(n+1)\left(-12-4(2n+1)+3n(n+1)\right)$ $n+\frac{1}{4}n(n+1)\left(3n^2-5n-16\right)$	**M1 A1**	Simplifies.
		6	

12 - (9231/13_Winter_2021_Q1)

	$b=-(\alpha+\beta+\gamma)=-3$	**B1**	
	$5=3^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1 A1**	Uses formula for sum of squares.
	$c=2$	**A1**	
	$6-3(5)+2(3)+3d=0$	**M1**	Uses original equation or formula for sum of cubes.
	[Equation is $x^3-3x^2+2x+1=0$] $d=1$	**A1**	
		6	

13 - (9231/11_Summer_2022_Q4)

(a)	$y = x^{-3} \Rightarrow x = y^{-\frac{1}{3}}$	**B1**	Substitutes.
	$\Rightarrow 2y^{-1} + 5y^{-\frac{2}{3}} - 6 = 0$ leading to $5y^{-\frac{2}{3}} = 6 - 2y^{-1} \Rightarrow 125y = (6y-2)^3$	**M1**	Cubes to eliminate radical.
	$216y^3 - 216y^2 - 53y - 8 = 0$	**A1**	
	Alternative method for question 4(a)		
	$(2x^3 - 6)^3 = (-5x^2)^3$	**M1**	$8x^9 - 72x^6 + 125x^6 + 216x^3 - 216 = 0$
	$y = x^{-3}$ leading to $x^3 = y^{-1}$	**B1**	Substitutes.
	$216y^3 - 216y^2 - 53y - 8 = 0$	**A1**	
		3	
(b)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = 1 \qquad \alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3} = -\frac{53}{216}$	**B1 FT**	Using *their* answer to part **(a)**.
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = 1^2 - 2\left(-\frac{53}{216}\right)$	**M1**	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = (\alpha^{-3} + \beta^{-3} + \gamma^{-3})^2 - 2(\alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3})$
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = \frac{161}{108}$	**A1**	
		3	
(c)	$216S_{-9} = 216S_{-6} + 53S_{-3} + 24$	**M1**	Using *their* $216\alpha^{-9} - 216\alpha^{-6} - 53\alpha^{-3} - 8 = 0$
	$S_{-9} = \frac{399}{216} = \frac{133}{72}$	**A1**	
		2	

14 - (9231/12_Summer_2022_Q4)

(a)	$y = x^{-3} \Rightarrow x = y^{-\frac{1}{3}}$	**B1**	Substitutes.
	$\Rightarrow 2y^{-1} + 5y^{-\frac{2}{3}} - 6 = 0$ leading to $5y^{-\frac{2}{3}} = 6 - 2y^{-1} \Rightarrow 125y = (6y-2)^3$	**M1**	Cubes to eliminate radical.
	$216y^3 - 216y^2 - 53y - 8 = 0$	**A1**	
	Alternative method for question 4(a)		
	$(2x^3 - 6)^3 = (-5x^2)^3$	**M1**	$8x^9 - 72x^6 + 125x^6 + 216x^3 - 216 = 0$
	$y = x^{-3}$ leading to $x^3 = y^{-1}$	**B1**	Substitutes.
	$216y^3 - 216y^2 - 53y - 8 = 0$	**A1**	
		3	
(b)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = 1 \qquad \alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3} = -\frac{53}{216}$	**B1 FT**	Using *their* answer to part **(a)**.
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = 1^2 - 2\left(-\frac{53}{216}\right)$	**M1**	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = (\alpha^{-3} + \beta^{-3} + \gamma^{-3})^2 - 2(\alpha^{-3}\beta^{-3} + \beta^{-3}\gamma^{-3} + \gamma^{-3}\alpha^{-3})$
	$\alpha^{-6} + \beta^{-6} + \gamma^{-6} = \frac{161}{108}$	**A1**	
		3	
(c)	$216S_{-9} = 216S_{-6} + 53S_{-3} + 24$	**M1**	Using *their* $216\alpha^{-9} - 216\alpha^{-6} - 53\alpha^{-3} - 8 = 0$
	$S_{-9} = \frac{399}{216} = \frac{133}{72}$	**A1**	
		2	

15 - (9231/11_Winter_2022_Q1)

(a)	$2\alpha+\gamma=-b$ $\alpha^2+2\alpha\gamma=0$	**B1**	
	$\alpha=-2\gamma$ leading to $-4\gamma+\gamma=-b$	**M1**	Solves simultaneous equations, or, express b and d in terms of α and γ.
	$\gamma=\frac{1}{3}b,\ \alpha=-\frac{2}{3}b$	**A1**	$b=3\gamma=-\dfrac{3}{2}\alpha$ and $d=-\alpha^2\gamma$.
	$\alpha^2\gamma=-d$ leading to $\frac{4}{27}b^3=-d$ leading to $4b^3+27d=0$	**M1 A1**	Substitutes into third equation, AG.
		5	
(b)	$3b=b^2$ leading to $b=3$	**M1 A1**	Uses $2\alpha^2+\gamma^2=(2\alpha+\gamma)^2-2(\alpha^2+2\alpha\gamma)$ or substitutes for α,γ in terms of b.
	$d=-4$	**A1**	
		3	

16 - (9231/12_Winter_2022_Q2)

(a)	$y = x^{-2}$ leading to $x = y^{-\frac{1}{2}}$	**B1**	Correct substitution.
	$y^{-2} + 3y^{-1} + 2y^{-\frac{1}{2}} + 6 = 0$ leading to $1 + 3y + 2y^{\frac{3}{2}} + 6y^2 = 0$	**M1**	Obtains an equation not involving radicals.
	$36y^4 + 32y^3 + 21y^2 + 6y + 1 = 0$	**A1**	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = -\frac{32}{36} = -\frac{8}{9}$	**B1 FT**	
		4	
(b)	$\alpha\beta\gamma\delta = 6$	**B1**	SOI
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	**M1**	Relates to coefficients.
	$\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 = -32$	**A1**	
		3	
(c)	$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} = \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2$ $-2\left(\alpha^{-2}\beta^{-2} + \alpha^{-2}\gamma^{-2} + a^{-2}\delta^{-2} + \beta^{-2}\gamma^{-2} + \beta^{-2}\delta^{-2} + \gamma^{-2}\delta^{-2}\right)$ $= \left(-\frac{8}{9}\right)^2 - 2\left(\frac{21}{36}\right)$	**M1**	Uses formula for sum of squares.
	$-\frac{61}{162}$	**A1**	
		2	

17 - (9231/13_Winter_2022_Q1)

(a)	$2\alpha+\gamma=-b$ $\alpha^2+2\alpha\gamma=0$	**B1**	
	$\alpha=-2\gamma$ leading to $-4\gamma+\gamma=-b$	**M1**	Solves simultaneous equations, or, express b and d in terms of α and γ.
	$\gamma=\frac{1}{3}b,\ \alpha=-\frac{2}{3}b$	**A1**	$b=3\gamma=-\dfrac{3}{2}\alpha$ and $d=-\alpha^2\gamma$.
	$\alpha^2\gamma=-d$ leading to $\frac{4}{27}b^3=-d$ leading to $4b^3+27d=0$	**M1 A1**	Substitutes into third equation, AG.
		5	
(b)	$3b=b^2$ leading to $b=3$	**M1 A1**	Uses $2\alpha^2+\gamma^2=(2\alpha+\gamma)^2-2(\alpha^2+2\alpha\gamma)$ or substitutes for α,γ in terms of b.
	$d=-4$	**A1**	
		3	

18 - (9231/11_Summer_2023_Q2)

(a)	$(-4)^2 - 2(6)$	**M1**	Uses formula for sum of squares.
	4	**A1**	
		2	
(b)	$(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = \sum_{r=1}^{n}\left(4 + 2(-4)r + 3r^2\right)$	**M1 A1**	Collects like terms and uses $\alpha + \beta + \lambda = -4$ and *their* result from part **(a)**.
	$4n - 4n(n+1) + \frac{1}{2}n(n+1)(2n+1)$	**M1**	Applies formulae from MF19.
	$-4n^2 + \frac{1}{2}n(n+1)(2n+1)$ $= n\left(-4n + \frac{1}{2}(2n^2 + 3n + 1)\right)$ $= n\left(n^2 - \frac{5}{2}n + \frac{1}{2}\right)$	**M1 A1**	Simplifies.
		6	

19 - (9231/12_Summer_2023_Q2)

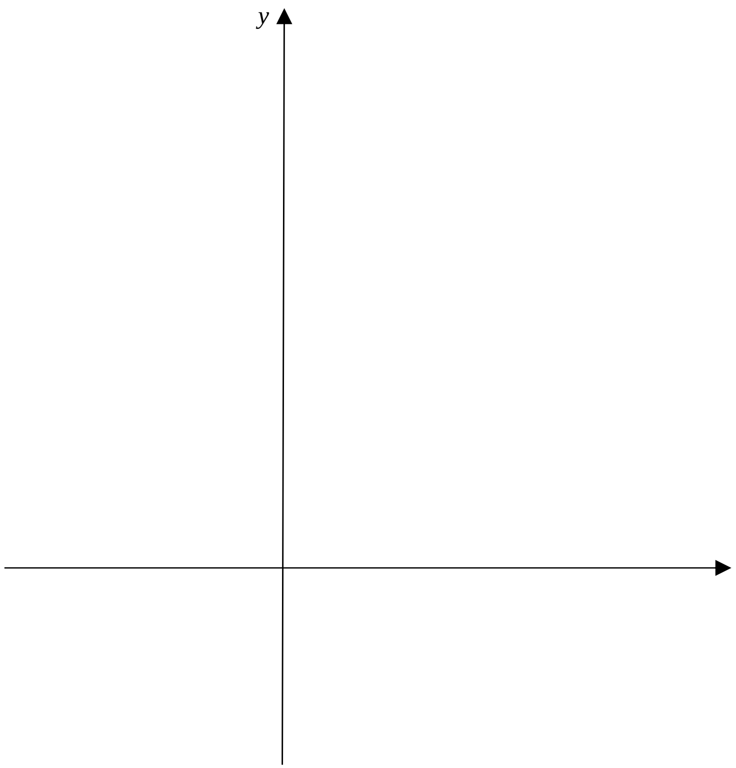
(a)	$(-4)^2 - 2(6)$	**M1**	Uses formula for sum of squares.
	4	**A1**	
		2	
(b)	$(\alpha + r)^2 = \alpha^2 + 2\alpha r + r^2$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = \sum_{r=1}^{n}\left(4 + 2(-4)r + 3r^2\right)$	**M1 A1**	Collects like terms and uses $\alpha + \beta + \lambda = -4$ and *their* result from part **(a)**.
	$4n - 4n(n+1) + \frac{1}{2}n(n+1)(2n+1)$	**M1**	Applies formulae from MF19.
	$-4n^2 + \frac{1}{2}n(n+1)(2n+1)$ $= n\left(-4n + \frac{1}{2}(2n^2 + 3n + 1)\right)$ $= n\left(n^2 - \frac{5}{2}n + \frac{1}{2}\right)$	**M1 A1**	Simplifies.
		6	

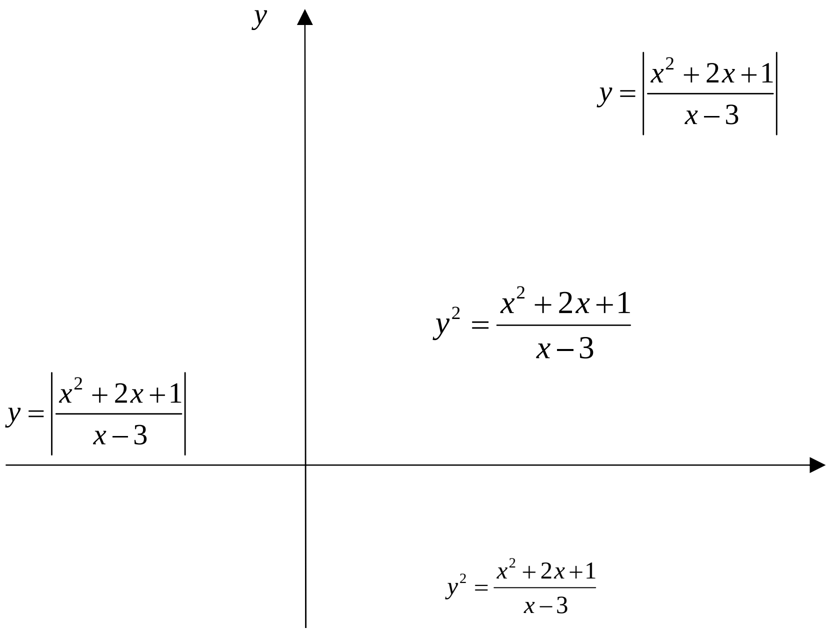
20 - (9231/13_Summer_2023_Q3)

(a)	$y = x^2$ so, $x = y^{\frac{1}{2}}$	**B1**	Correct substitution.
	$y^2 - y + 2y^{\frac{1}{2}} + 5 = 0$ so, $\left(2y^{\frac{1}{2}}\right)^2 = \left(-y^2 + y - 5\right)^2$	**M1**	Obtains an equation which eliminates radicals.
	$y^4 - 2y^3 + 11y^2 - 14y + 25 = 0$	**A1**	
	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 2$	**B1FT**	
		4	
(b)	$\alpha^2\beta^2\gamma^2\delta^2 = 25$	**B1FT**	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	**M1**	Relates to coefficients.
	$\frac{14}{25}$	**A1**	CAO
		3	
(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = \left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right)^2$ $- 2\left(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2\right)$ $= 2^2 - 2(11)$	**M1**	Uses formula for sum of squares or uses original equation.
	-18	**A1**	
		2	

21 - (9231/13_Summer_2023_Q7)

(a)	$x=3$	**B1**	States vertical asymptote.
	$y=\frac{(x-3)(x+5)+16}{x-3}=x+5+\frac{16}{x-3}$	**M1**	Finds oblique asymptote.
	$y=x+5$	**A1**	
		3	
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{16}{(x-3)^2}=0\Rightarrow(x-3)^2=16$	**M1**	Differentiates and sets equal to zero.
	$x=-1,7$	**A1**	Finds *x*-coordinates
	$(-1,0),(7,16)$	**A1**	States coordinates of turning points.
		3	

(c)		**B1FT**	Axes and labelled asymptotes.
		B1	Upper branch correct.
		B1	Lower branch correct and no additional branches.
		3	

(d)		**B1FT**	Clear labels, axes and their vertical asymptote.
		B1FT	$y=\left\lvert\frac{x^2+2x+1}{x-3}\right\rvert$ correct, FT from their sketch in **(c)**.
		B1	Upper branch of $y^2=\frac{x^2+2x+1}{x-3}$ (positive square root).
		B1FT	Lower branch of $y^2=\frac{x^2+2x+1}{x-3}$ (negative square root). FT from previous mark.
		4	

22 - (9231/11_Winter_2023_Q3)

(a)	$b = -(\alpha+\beta+\gamma+\delta) = -3$	**B1**	
	$5 = (-3)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	**M1 A1**	Uses formula for sum of squares.
	$c = 2$	**A1**	
	$6 = \dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta} = \dfrac{-d}{-2}$	**M1**	Uses $\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1} = \dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$.
	$d = 12$	**A1**	Equation is $x^4 - 3x^3 + 2x^2 + 12x - 2 = 0$.
		6	
(b)	$\alpha^4+\beta^4+\gamma^4+\delta^4 = 3(-27) - 2(5) - 12(3) + 2(4)$	**M1**	Uses *their* quartic equation derived in **(a)**.
	-119	**A1**	
		2	

23 - (9231/12_Winter_2023_Q4)

(a)	$y = 3x+1 \Rightarrow x = \frac{1}{3}(y-1)$ $\Rightarrow 27\left(\frac{y-1}{3}\right)^3 + 18\left(\frac{y-1}{3}\right)^2 + 6\left(\frac{y-1}{3}\right) - 1 = 0$	**B1**	Substitutes.
	$\Rightarrow (y-1)^3 + 2(y-1)^2 + 2(y-1) - 1 = 0$ $\Rightarrow y^3 - 3y^2 + 3y - 1 + 2y^2 - 4y + 2 + 2y - 2 - 1 = 0$	**M1**	Expands.
	$y^3 - y^2 + y - 2 = 0$	**A1**	AG.
		3	
(b)	$S_2 = 1^2 - 2(1) = -1$	**M1 A1**	Uses formula for sum of squares, AG.
	$S_3 = (3\alpha+1)^3 + (3\beta+1)^3 + (3\gamma+1)^3 = -1-(1)+6$	**M1**	Uses $y^3 = y^2 - y + 2$ or expands and uses original equation.
	4	**A1**	
		4	
(c)	$S_{-1} = \frac{(3\alpha+1)(3\beta+1)+(3\beta+1)(3\gamma+1)+(3\gamma+1)(3\alpha+1)}{(3\alpha+1)(3\beta+1)(3\gamma+1)} = \frac{1}{2}$	**B1**	
	$2S_{-2} = S_1 - 3 + S_{-1} = 1 - 3 + \frac{1}{2}$	**M1**	Uses $2y^{-2} = y - 1 + y^{-1}$.
	$S_{-2} = -\frac{3}{4}$	**A1**	CAO
		3	

24 - (9231/13_Winter_2023_Q3)

(a)	$b=-(\alpha+\beta+\gamma+\delta)=-3$	**B1**	
	$5=(-3)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	**M1 A1**	Uses formula for sum of squares.
	$c=2$	**A1**	
	$6=\dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}=\dfrac{-d}{-2}$	**M1**	Uses $\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}=\dfrac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$.
	$d=12$	**A1**	Equation is $x^4-3x^3+2x^2+12x-2=0$.
		6	
(b)	$\alpha^4+\beta^4+\gamma^4+\delta^4=3(-27)-2(5)-12(3)+2(4)$	**M1**	Uses *their* quartic equation derived in **(a)**.
	-119	**A1**	
		2	

25 - (9231/11_Summer_2024_Q1)

(a)	$-\frac{1}{2}p$	**B1**	
		1	
(b)	$\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta\gamma^2=\alpha\beta\gamma(\beta+\gamma+\alpha)=\frac{5}{2}\left(-\frac{1}{2}\right)$	**M1**	Factorises.
	$-\frac{5}{4}$	**A1**	
		2	
(c)	$4z^3+2pz^2-5z-25=0$	**B1 FT**	OE FT on *their* value of $\alpha\beta+\beta\gamma+\gamma\alpha$ (in terms of p).
		1	
(d)	$\frac{1}{3}=\frac{1}{4}+p$	**M1**	Uses $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
	$p=\frac{1}{12}$	**A1**	CAO
		2	

26 - (9231/12_Summer_2024_Q1)

(a)	$-\frac{1}{2}p$	**B1**	
		1	
(b)	$\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta\gamma^2=\alpha\beta\gamma\left(\beta+\gamma+\alpha\right)=\frac{5}{2}\left(-\frac{1}{2}\right)$	**M1**	Factorises.
	$-\frac{5}{4}$	**A1**	
		2	
(c)	$4z^3+2pz^2-5z-25=0$	**B1 FT**	OE FT on *their* value of $\alpha\beta+\beta\gamma+\gamma\alpha$ (in terms of *p*).
		1	
(d)	$\frac{1}{3}=\frac{1}{4}+p$	**M1**	Uses $\alpha^2+\beta^2+\gamma^2=\left(\alpha+\beta+\gamma\right)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
	$p=\frac{1}{12}$	**A1**	CAO
		2	

27 - (9231/13_Summer_2024_Q2)

(a)	$y = x^2 + 1$	**B1**	Uses correct substitution. May use a different letter.
	$(y-1)^{\frac{3}{2}} + 2(y-1) + 3(y-1)^{\frac{1}{2}} + 1 = 0 \Rightarrow (y-1)^{\frac{1}{2}}((y-1)+3) = -2(y-1) - 1$ $(y-1)(y+2)^2 = (-2y+1)^2 \Rightarrow (y-1)(y^2+4y+4) = 4y^2 - 4y + 1$ OR $(x^3+3x)^2 = (-2x^2-1)^2 \Rightarrow x^6 + 2x^4 + 5x^2 - 1 = 0$ $(y-1)^3 + 2(y-1)^2 + 5(y-1) - 1 = 0$	**M1**	Substitutes and obtains an equation not involving radicals.
	$y^3 - y^2 + 4y - 5 = 0$	**A1**	OE
		3	
(b)	$(\alpha^2+1)^2 + (\beta^2+1)^2 + (\gamma^2+1)^2 = 1 - 2(4)$	**M1**	Uses $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2$ $-2(\alpha\beta + \beta\gamma + \gamma\alpha)$. on *their* answer for part **2(a)**.
	-7	**A1**	CAO
		2	

(c)	$(\alpha^2+1)^3+(\beta^2+1)^3+(\gamma^2+1)^3=-7-4+5(3)$	**M1**	Uses *their* equation from part **2(a)**.
	4	**A1**	CAO
	Alternative method for question 2(c)		
	Uses $\sum(\alpha^2+1)^3=$ $\left(\sum(\alpha^2+1)\right)^3-3\left(\sum(\alpha^2+1)\right)\left(\sum(\alpha^2+1)(\beta^2+1)\right)+3\left((\alpha^2+1)(\beta^2+1)(\gamma^2+1)\right)$ $=1^3-3\times1\times4+3\times5$	**M1**	Uses *their* equation from part **2(a)** in a valid formula.
	4	**A1**	CAO
		2	

28 - (9231/11_Winter_2024_Q3)

(a)	$y = x^4$	**B1**	Uses correct substitution.
	$y + 2y^{\frac{3}{4}} - 1 = 0 \Rightarrow 16y^3 = (1-y)^4$	**M1**	Substitutes and obtains an equation not involving radicals.
	$16y^3 = 1 - 4y + 6y^2 - 4y^3 + y^4$	**M1**	Uses binomial expansion.
	$y^4 - 20y^3 + 6y^2 - 4y + 1 = 0$	**A1**	Must be an equation.
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 20$	**B1**	
		5	
(b)	$\alpha + \beta + \gamma + \delta = -2$	**B1**	
	$x^5 + 2x^4 - x = 0 \Rightarrow \alpha^5 + \beta^5 + \gamma^5 + \delta^5 = -2(20) + (-2)$	**M1**	Multiplies original equation by x and substitutes.
	-42	**A1**	
		3	
(c)	$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 = 20^2 - 2(6)$	**M1**	Uses formula for sum of squares. Or alternative complete method eg another substitution.
	388	**A1**	CAO.
		2	

29 - (9231/12_Winter_2024_Q3)

(a)	$\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$ $3=2^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	**M1**	Substitutes into formula for sum of squares.
	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{1}{2}$	**A1**	
		2	
(b)	$\alpha^2(\beta+\gamma+\delta)+\beta^2(\alpha+\gamma+\delta)+\gamma^2(\alpha+\beta+\delta)+\delta^2(\alpha+\beta+\gamma)$ $=\alpha^2(2-\alpha)+\beta^2(2-\beta)+\gamma^2(2-\gamma)+\delta^2(2-\delta)$	**M1 A1**	Factorises and substitutes.
	$2(3)-4=2$	**A1**	
		3	
(c)(i)	$6S_4-12S_3+3S_2+2S_1+24=0\Rightarrow 6S_4-12(4)+3(3)+2(2)+24=0$	**M1**	Sums and substitutes.
		A1	
	$S_4=\frac{11}{6}$	**A1**	
		3	
(c)(ii)	$6S_5-12S_4+3S_3+2S_2+6S_1=0\Rightarrow 6S_5-12\left(\frac{11}{6}\right)+3(4)+2(3)+12=0$	**M1**	Multiplies equation through by x, sums and substitutes. $S_5-2S_4+\frac{1}{2}S_3+\frac{1}{3}S_2+S_1=0$ $\Rightarrow S_5-2\left(\frac{11}{6}\right)+\frac{1}{2}(4)+\frac{1}{3}(3)+2=0$
	$S_5=-\frac{4}{3}$	**A1**	
		2	

30 - (9231/13_Winter_2024_Q3)

(a)	$y = x^4$	**B1**	Uses correct substitution.
	$y + 2y^{\frac{3}{4}} - 1 = 0 \Rightarrow 16y^3 = (1-y)^4$	**M1**	Substitutes and obtains an equation not involving radicals.
	$16y^3 = 1 - 4y + 6y^2 - 4y^3 + y^4$	**M1**	Uses binomial expansion.
	$y^4 - 20y^3 + 6y^2 - 4y + 1 = 0$	**A1**	Must be an equation.
	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 20$	**B1**	
		5	
(b)	$\alpha + \beta + \gamma + \delta = -2$	**B1**	
	$x^5 + 2x^4 - x = 0 \Rightarrow \alpha^5 + \beta^5 + \gamma^5 + \delta^5 = -2(20) + (-2)$	**M1**	Multiplies original equation by x and substitutes.
	-42	**A1**	
		3	
(c)	$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 = 20^2 - 2(6)$	**M1**	Uses formula for sum of squares. Or alternative complete method eg another substitution.
	388	**A1**	CAO.
		2	

31 - (9231/21_Winter_2024_Q4)

(a)	$\begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$	**M1**	Multiplies matrix with eigenvector.
	$\lambda = -2$	**A1**	
		2	
(b)	$\begin{vmatrix} -11-\lambda & 1 & 8 \\ 0 & -2-\lambda & 0 \\ -16 & 1 & 13-\lambda \end{vmatrix} = 0$	**M1**	Sets determinant equal to zero.
	$(-11-\lambda)(-2-\lambda)(13-\lambda)+128(-2-\lambda)=0$ $(-2-\lambda)(\lambda^2-2\lambda-15)=0$ $\lambda^3-19\lambda-30=0$	**A1**	Expands determinant, AG.
	$\lambda = 5, -3$	**B1**	
		3	
(c)	$\mathbf{A}^3 - 19\mathbf{A} - 30\mathbf{I} = 0$	**B1**	States that **A** satisfies its characteristic equation.
	$30\mathbf{A}^{-1} = \mathbf{A}^2 - 19\mathbf{I}$	**M1**	Multiplies through by $\mathbf{A}^{-1}$.
	$\mathbf{A}^2 = \begin{pmatrix} -7 & -5 & 16 \\ 0 & 4 & 0 \\ -32 & -5 & 41 \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{30}\begin{pmatrix} -26 & -5 & 16 \\ 0 & -15 & 0 \\ -32 & -5 & 22 \end{pmatrix}$	**M1A1**	M1 for substituting $\mathbf{A}^2$.
		4	

32 - (9231/22_Winter_2024_Q4)

(a)	$\cos 6\theta = \text{Re}(\cos\theta + \text{i}\sin\theta)^6 = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$	**M1A1**	Expands and takes real part.
	$\sin 6\theta = \text{Im}(\cos\theta + \text{i}\sin\theta)^6 = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$	**M1A1**	Takes imaginary part
	$\cot 6\theta = \dfrac{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta} \times \dfrac{\sin^{-6}\theta}{\sin^{-6}\theta}$	**M1**	Divides numerator and denominator by $\sin^6\theta$.
	$\cot 6\theta = \dfrac{\cot^6\theta - 15\cot^4\theta + 15\cot^2\theta - 1}{6\cot^5\theta - 20\cot^3\theta + 6\cot\theta}$	**A1**	AG
		6	
(b)	$x = \cot\theta, \quad \cot 6\theta = 1$	**B1**	Applies identity given in **(b)**.
	$6\theta = \frac{1}{4}\pi + k\pi$	**M1**	Solves $\cot 6\theta = 1$
	$\cot\left(\frac{1}{24}\pi\right)$	**A1**	Gives one correct solution. 7.60
	$\cot\left(\frac{5}{24}\pi\right), \cot\left(\frac{3}{8}\pi\right), \cot\left(\frac{13}{24}\pi\right), \cot\left(\frac{17}{24}\pi\right), \cot\left(\frac{7}{8}\pi\right)$	**A1**	Gives other five solutions. 1.30, 0.41, –0.13, –0.77, –2.41 Withhold A1 mark for repeated roots.
		4	

33- (9231/23_Winter_2024_Q1)

	$\begin{vmatrix} 1 & 5 & 6 \\ k & 2 & 2 \\ -3 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 4 & 8 \end{vmatrix} - 5\begin{vmatrix} k & 2 \\ -3 & 8 \end{vmatrix} + 6\begin{vmatrix} k & 2 \\ -3 & 4 \end{vmatrix} = 8 - 5(8k+6) + 6(4k+6)$	**M1A1**	Evaluates determinant.
	$k \neq \frac{7}{8}$	**A1**	
	Three planes intersect at a *unique* point.	**B1**	
		4	

34 - (9231/23_Winter_2024_Q4)

(a)	$\begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$	**M1**	Multiplies matrix with eigenvector.
	$\lambda = -2$	**A1**	
		2	

(b)	$\begin{vmatrix} -11-\lambda & 1 & 8 \\ 0 & -2-\lambda & 0 \\ -16 & 1 & 13-\lambda \end{vmatrix} = 0$	**M1**	Sets determinant equal to zero.
	$(-11-\lambda)(-2-\lambda)(13-\lambda)+128(-2-\lambda)=0$ $(-2-\lambda)(\lambda^2-2\lambda-15)=0$ $\lambda^3-19\lambda-30=0$	**A1**	Expands determinant, AG.
	$\lambda = 5, -3$	**B1**	
		3	
(c)	$\mathbf{A}^3 - 19\mathbf{A} - 30\mathbf{I} = 0$	**B1**	States that **A** satisfies its characteristic equation.
	$30\mathbf{A}^{-1} = \mathbf{A}^2 - 19\mathbf{I}$	**M1**	Multiplies through by $\mathbf{A}^{-1}$.
	$\mathbf{A}^2 = \begin{pmatrix} -7 & -5 & 16 \\ 0 & 4 & 0 \\ -32 & -5 & 41 \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{30}\begin{pmatrix} -26 & -5 & 16 \\ 0 & -15 & 0 \\ -32 & -5 & 22 \end{pmatrix}$	**M1A1**	M1 for substituting $\mathbf{A}^2$.
		4	

35 - (9231/11_Summer_2025_Q2)

(a)	$y = x^3 - 1$	B1	Uses correct substitution.
	$(y+1)^{\frac{3}{3}} + 2(y+1)^{\frac{1}{3}} + 1 = 0 \Rightarrow 2(y+1)^{\frac{1}{3}} = -y - 2$ $8(y+1) = -(y+2)^3 \Rightarrow 8y + 8 = -y^3 - 6y^2 - 12y - 8$	M1	Substitutes and obtains an equation not involving radicals.
	$y^3 + 6y^2 + 20y + 16 = 0$	A1	
		3	
(b)	$(\alpha^3 - 1)^2 + (\beta^3 - 1)^2 + (\gamma^3 - 1)^2 = 6^2 - 2(20)$	M1	Uses $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$.
	-4	A1	FT
		2	
(c)	$(\alpha^3 - 1)^3 + (\beta^3 - 1)^3 + (\gamma^3 - 1)^3 = -6(-4) - 20(-6) - 16(3)$	M1	Uses their equation from (a) or a valid formula for the sum of cubes.
	96	A1	
		2	

36 - (9231/12_Summer_2025_Q2)

(a)	$y = x^3 - 1$	**B1**	Uses correct substitution.
	$(y+1)^{\frac{3}{3}} + 2(y+1)^{\frac{1}{3}} + 1 = 0 \Rightarrow 2(y+1)^{\frac{1}{3}} = -y-2$ $8(y+1) = -(y+2)^3 \Rightarrow 8y+8 = -y^3 - 6y^2 - 12y - 8$	**M1**	Substitutes and obtains an equation not involving radicals.
	$y^3 + 6y^2 + 20y + 16 = 0$	**A1**	
		3	
(b)	$(\alpha^3-1)^2 + (\beta^3-1)^2 + (\gamma^3-1)^2 = 6^2 - 2(20)$	**M1**	Uses $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$.
	-4	**A1**	FT
		2	
(c)	$(\alpha^3-1)^3 + (\beta^3-1)^3 + (\gamma^3-1)^3 = -6(-4) - 20(-6) - 16(3)$	**M1**	Uses their equation from (a) or a valid formula for the sum of cubes.
	96	**A1**	
		2	

37 - (9231/13_Summer_2025_Q3)

(a)	$0=\alpha^2+\beta^2+\gamma^2+\delta^2+2(7)$	**M1**	Uses $(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+$ $2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
	$\alpha^2+\beta^2+\gamma^2+\delta^2=-14$	**A1**	
		2	
(b)	$S_4+7S_2+3S_1+88=0$	**M1**	Uses given equation.
	$S_4=10$	**A1FT**	FT from **(a)**
		2	
(c)	$(r+\alpha^2)^2=r^2+2\alpha^2r+\alpha^4$	**B1**	Expands.
	$\sum_{r=1}^{10}\left((r+\alpha^2)^2+(r+\beta^2)^2+(r+\gamma^2)^2+(r+\delta^2)^2\right)$ $=\sum_{r=1}^{10}(4r^2-28r+10)$	**M1**	Collects like terms and uses results from parts **(a)** and **(b)**.
		A1FT	FT from **(a)** and **(b)** evaluated coefficients.
	$=4\left(\frac{10}{6}(10+1)(2(10)+1)\right)-14(10)(10+1)$ $+10(10)$	**M1**	Applies formulae from MF19.
	$=100$	**A1**	Evaluated. CAO
		5	

38 - (9231/14_Summer_2025_Q4)

(a)	$\begin{vmatrix}1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1\end{vmatrix} = \begin{vmatrix}1 & \gamma \\ \gamma & 1\end{vmatrix} - \alpha\begin{vmatrix}\alpha & \gamma \\ \beta & 1\end{vmatrix} + \beta\begin{vmatrix}\alpha & 1 \\ \beta & \gamma\end{vmatrix} = 1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma$	**M1 A1**	Finds determinant.
	$1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma=0 \Rightarrow \alpha^2+\beta^2+\gamma^2=1+2(1)=3$	**M1 A1**	Sets determinant $=0$ and uses $\alpha\beta\gamma=1$, AG.
		4	
(b)	$\alpha^2+\beta^2+\gamma^2=b^2-2c \Rightarrow 3=b^2-2c$	***M1 A1**	Uses $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$.
	$\alpha^3+\beta^3+\gamma^3=-b(3)-c(-b)+3 \Rightarrow 3=-3b+cb+3$	***M1 A1**	Uses cubic equation or formula for $\alpha^3+\beta^3+\gamma^3$.
	$c=3, \quad b=3$	**DM1 A1**	Solves simultaneous equations. WWW
		6	

39- (9231/21_Summer_2025_Q1)

	$z^3 = 27 - 27\mathrm{i} = 27\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi}$	**B1**	Modulus and argument of z^3 (does not need to be in exponential form).
	$z_1 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\frac{1}{12}\pi}$	**M1 A1**	Finds one root. Cube root must be taken for M1.
	$z_2 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{7}{12}\pi}$, $z_3 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\frac{9}{12}\pi}$	**A1FT A1FT**	Finds other two roots. FT on modulus. Withhold one answer mark for omission of i.
		5	

40 - (9231/23_Summer_2025_Q5)

(a)	$\left(\cos 5\theta + \mathrm{i}\sin 5\theta\right) = \left(c + \mathrm{i}s\right)^5$	**M1**	Uses binomial theorem.
	$\cos 5\theta = c^5 - 10s^2c^3 + 5s^4c$	**A1**	
	$= c^5 - 10\left(1-c^2\right)c^3 + 5\left(1-c^2\right)^2 c$	**M1**	Uses $s^2 = 1 - c^2$.
	$= 16c^5 - 20c^3 + 5c$	**A1**	
	$\sec 5\theta = \dfrac{1}{16c^5 - 20c^3 + 5c} \times \dfrac{\sec^5\theta}{\sec^5\theta}$	**M1**	Divides *simplified* numerator and denominator by c^5.
	$= \dfrac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}$	**A1**	AG.
		6	
(b)	$x^5 = \frac{2}{\sqrt{3}}\left(5x^4 - 20x^2 + 16\right) \Rightarrow \dfrac{x^5}{5x^4 - 20x^2 + 16} = \frac{2}{\sqrt{3}}$	**B1**	Relates with equation in part (a). Can be implied by M1 mark.
	$\sec 5\theta = \frac{2}{\sqrt{3}} \Rightarrow \cos 5\theta = \frac{\sqrt{3}}{2}$	**M1**	Solves $\cos 5\theta = \frac{\sqrt{3}}{2}$.
	$x = \sec\left(\frac{1}{30}\pi\right)$	**A1**	Gives one correct solution.
	$\sec\left(\frac{11}{30}\pi\right), \sec\left(\frac{13}{30}\pi\right), \sec\left(\frac{23}{30}\pi\right), \sec\left(\frac{25}{30}\pi\right)$	**A1**	Gives four other solutions. Allow different values of q as long as all five solutions are found. -1.35, -1.15, 1.01, 2.46, 4.81
		4	

41 - (9231/24_Summer_2025_Q5)

(a)	$\sin 7\theta = \text{Im}\left((c+\text{i}s)^7\right)$	**M1**	Uses binomial expansion and takes imaginary part.
	$= -s^7 + 21c^2s^5 - 35c^4s^3 + 7c^6s$	**A1**	
	$= -s^7 + 21\left(1-s^2\right)s^5 - 35\left(1-s^2\right)^2 s^3 + 7\left(1-s^2\right)^3 s$	**M1**	Applies $c^2 = 1-s^2$.
	$\left(1-s^2\right)^3 = -s^6 + 3s^4 - 3s^2 + 1$	**B1**	No sight of $\left(1-s^2\right)^3$ *fully* expanded is B0.
	$\sin 7\theta = -64s^7 + 112s^5 - 56s^3 + 7s$	**A1**	AG.
		5	
(b)	$x = \sin\theta \quad \sin\theta \neq 0 \quad \sin 7\theta = 0$	**M1**	Solves $\sin 7\theta = 0$ and excludes $\sin\theta = 0$. For M1 the roots from $\sin\theta = 0$ must be excluded at some stage of their working. (This could be implied by their final answers.)
	$x = \sin\left(\frac{1}{7}\pi\right)$	**A1**	One correct root.
	$\sin\left(-\frac{1}{7}\pi\right), \quad \sin\left(\pm\frac{2}{7}\pi\right), \quad \sin\left(\pm\frac{3}{7}\pi\right)$	**A1**	Exactly five other correct roots. Allow different values of q which give these roots. Accept $-\sin\left(\frac{1}{7}\pi\right), \quad \pm\sin\left(\frac{2}{7}\pi\right), \quad \pm\sin\left(\frac{3}{7}\pi\right)$. Just listing correct values of q without any reference of x is M1 SCB1 (2/3).
		3	

42 - (9231/11_Winter_2025_Q4)

(a)	$y=2x+1 \Rightarrow x=\frac{1}{2}(y-1)$	**B1**	Correct substitution used in equation.
	$\frac{1}{16}(y-1)^4+\frac{1}{8}(y-1)^3+\frac{1}{4}(y-1)^2+\frac{1}{2}(y-1)+1=0$ $\frac{1}{16}(y^4-4y^3+6y^2-4y+1)+\frac{1}{8}(y^3-3y^2+3y-1)+\frac{1}{4}(y^2-2y+1)+\frac{1}{2}(y-1)+1=0$	**M1 A1**	Expands.
	$\frac{1}{16}y^4-\frac{1}{8}y^3+\frac{1}{4}y^2+\frac{1}{8}y+\frac{11}{16}=0 \Rightarrow y^4-2y^3+4y^2+2y+11=0$	**A1**	Simplifies, AG.
		4	

(b)	$S_2=2^2-2(4)$	**M1**	Uses $(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2$ $+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
	$S_2=-4$	**A1**	
		2	
(c)	$S_4-2S_3+4S_2+2S_1+44=0$	**M1**	Sums over roots.
	$S_4=-76$	**A1**	
		2	

43 - (9231/12_Winter_2025_Q2)

(a)	$b = -(\alpha + \beta + \gamma) = -2$	**B1**	
	$3 = (-2)^2 - 2c$	**M1**	Uses formula for sum of squares.
	$c = \frac{1}{2}$	**A1**	
		3	
(b)	$S_3 + bS_2 + cS_1 + 3d = 0$	**M1**	Uses original equation or formula for sum of cubes.
	$S_3 = -3b - 2c - 3d = 5 - 3d$	**A1FT**	Allow FT from part **(a)** for their value.
	$S_4 + bS_3 + cS_2 + dS_1 = 0$	**M1**	Uses original equation.
	$5 + (-2)(5 - 3d) + \left(\frac{1}{2}\right)(3) + 2d = 0 \Rightarrow 8d - \frac{7}{2} = 0$	**A1FT**	Allow FT from part **(a)** for their value and for their value of S_3.
	$d = \frac{7}{16}$	**A1**	CAO
		5	

44 - (9231/14_Winter_2025_Q2)

(a)	$\alpha = -d$	**B1**	SOI
	$\alpha + \beta + \frac{1}{\beta} = -b$	**M1**	Uses $\alpha + \beta + \gamma = -b$ and replaces γ with $\frac{1}{\beta}$. OE
	$\beta + \frac{1}{\beta} = d - b$	**A1**	AG.
		3	
(b)	$-d\beta + 1 - \frac{d}{\beta} = c$	**M1**	Uses $\alpha\beta + \beta\gamma + \gamma\alpha = c$.
	$\beta + \frac{1}{\beta} = \frac{1-c}{d}$	**A1**	AG.
		2	
(c)(i)	$d - 3 = \frac{1+3}{d} \Rightarrow d^2 - 3d - 4 = 0$	**M1**	Equates expressions given in **(a)** and **(b)**, with correct use of coefficients.
	$d = 4$	**A1**	$d = 4$ only for final answer.
		2	
(c)(ii)	$\alpha^2 + \beta^2 + \gamma^2 = b^2 - 2c$	**M1**	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ with correct use of coefficients.
	15	**A1**	
		2	